MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.14 Analysis and Design of Feedback Control Systems

Introduction to Frequency Domain Processing

1 Introduction - Superposition

In this set of notes we examine an alternative to the time-domain convolution operations describing the input-output operations of a linear processing system. The methods developed here use Fourier techniques to transform the temporal representation $f(t)$ to a reciprocal frequency domain space $F(j\omega)$ where the difficult operation of convolution is replaced by simple multiplication. In addition, an understanding of Fourier methods gives qualitative insights to signal processing techniques such as filtering.

Linear systems, by definition, obey the *principle of superposition* for the forced component of their responses:

> If linear system is at rest at time $t = 0$, and is subjected to an input $u(t)$ that is the sum of a set of causal inputs, that is $u(t) = u_1(t) + u_2(t) + \ldots$, the response $y(t)$ will be the sum of the individual responses to each component of the input, that is $y(t) = y_1(t) + y_2(t) + \ldots$

Suppose that a system input $u(t)$ may be expressed as a sum of complex n exponentials

$$u(t) = \sum_{i=1}^{n} a_i e^{s_i t},$$

where the complex coefficients a_i and constants s_i are known. Assume that each component is applied to the system alone; if at time $t = 0$ the system is at rest, the solution component $y_i(t)$ is of the form

$$y_i(t) = \left(y_h\left(t\right)\right)_i + a_i H(s_i) e^{s_i t}$$

where $(y_h(t))_i$ is a homogeneous solution. The principle of superposition states that the total response $y_p(t)$ of the linear system is the sum of all component outputs

$$\begin{aligned} y_p(t) &= \sum_{i=1}^{n} y_i(t) \\ &= \sum_{i=1}^{n} \left(y_h\left(t\right)\right)_i + a_i H(s_i) e^{s_i t} \end{aligned}$$

■ Example

Find the response of the first-order system with differential equation

$$\frac{dy}{dt} + 4y = 2u$$

to an input $u(t) = 5e^{-t} + 3e^{-2t}$, given that at time $t = 0$ the response is $y(0) = 0$.

Solution: The system transfer function is

$$H(s) = \frac{2}{s+4}. \tag{1}$$

The system homogeneous response is

$$y_h(t) = Ce^{-4t} \tag{2}$$

where C is a constant, and if the system is at rest at time $t = 0$, the response to an exponential input $u(t) = ae^{st}$ is

$$y(t) = \frac{2a}{s+4}\left(e^{st} - e^{-4t}\right). \tag{3}$$

The principle of superposition says that the response to the input $u(t) = 5e^{-t} + 3e^{-2t}$ is the sum of two components, each similar to Eq.(3), that is

$$\begin{aligned} y(t) &= \frac{10}{3}\left(e^{-t} - e^{-4t}\right) + \frac{6}{2}\left(e^{-2t} - e^{-4t}\right) \\ &= \frac{10}{3}e^{-t} + 3e^{-2t} - \frac{19}{3}e^{-4t} \end{aligned} \tag{4}$$

In this chapter we examine methods that allow a function of time $f(t)$ to be represented as a sum of elementary sinusoidal or complex exponential functions. We then show how the system transfer function $H(s)$, or the frequency response $H(j\omega)$, defines the response to each such component, and through the principle of superposition defines the total response. These methods allow the computation of the response to a very broad range of input waveforms, including most of the system inputs encountered in engineering practice.

The methods are known collectively as Fourier Analysis methods, after Jean Baptiste Joseph Fourier, who in the early part of the 19th century proposed that an arbitrary repetitive function could be written as an infinite sum of sine and cosine functions [1]. The Fourier Series representation of periodic functions may be extended through the Fourier Transform to represent non-repeating aperiodic (or transient) functions as a continuous distribution of sinusoidal components. A further generalization produces the Laplace Transform representation of waveforms.

The methods of representing and analyzing waveforms and system responses in terms of the action of the frequency response function on component sinusoidal or exponential waveforms are known collectively as the *frequency-domain* methods. Such methods, developed in this chapter, have important theoretical and practical applications throughout engineering, especially in system dynamics and control system theory.

2 Fourier Analysis of Periodic Waveforms

Consider the steady-state response of linear time-invariant systems to two *periodic* waveforms, the real sinusoid $f(t) = \sin\omega t$ and the complex exponential $f(t) = e^{j\omega t}$. Both functions are repetitive; that is they have identical values at intervals in time of $t = 2\pi/\omega$ seconds apart. In general a *periodic function* is a function that satisfies the relationship:

$$f(t) = f(t+T) \tag{5}$$

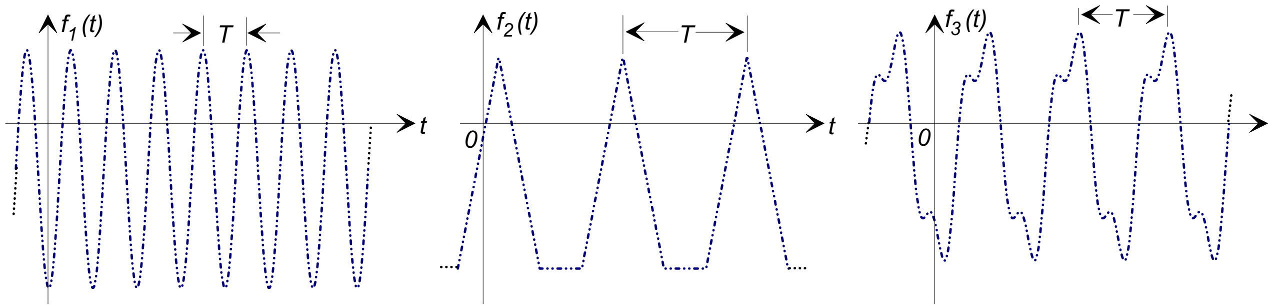

Figure 1: Examples of periodic functions of time.

for all t, or $f(t) = f(t + nT)$ for $n = \pm 1, \pm 2, \pm 3, ...$ Figure 1 shows some examples of periodic functions.

The *fundamental angular frequency* ω_0 (in radians/second) of a periodic waveform is defined directly from the period

$$\omega_0 = \frac{2\pi}{T}. \tag{6}$$

Any periodic function with period T is also be periodic at intervals of nT for any positive integer n. Similarly any waveform with a period of T/n is periodic at intervals of T seconds. Two waveforms whose periods, or frequencies, are related by a simple integer ratio are said to be *harmonically related*.

Consider, for example, a pair of periodic functions; the first $f_1(t)$ with a period of $T_1 = 12$ seconds, and the second $f_2(t)$ with a period of $T_2 = 4$ seconds. If the fundamental frequency ω_0 is defined by $f_1(t)$, that is $\omega_o = 2\pi/12$, then $f_2(t)$ has a frequency of $3\omega_0$. The two functions are harmonically related, and $f_2(t)$ is said to have a frequency which is the *third* harmonic of the fundamental ω_0. If these two functions are summed together to produce a new function $g(t) = f_1(t) + f_2(t)$, then $g(t)$ will repeat at intervals defined by the longest period of the two, in this case every 12 seconds. In general, when harmonically related waveforms are added together the resulting function is also periodic with a repetition period equal to the fundamental period.

■ Example

A family of waveforms $g_N(t)$ $(N = 1, 2 \ldots 5)$ is formed by adding together the first N of up to five component functions, that is

$$g_N(t) = \sum_{n=1}^{N} f_n(t) \qquad\qquad 1 < N \leq 5$$

where

$$\begin{aligned} f_1(t) &= 1 \\ f_2(t) &= sin(2\pi t) \\ f_3(t) &= \frac{1}{3} sin(6\pi t) \\ f_4(t) &= \frac{1}{5} sin(10\pi t) \\ f_5(t) &= \frac{1}{7} sin(14\pi t). \end{aligned}$$

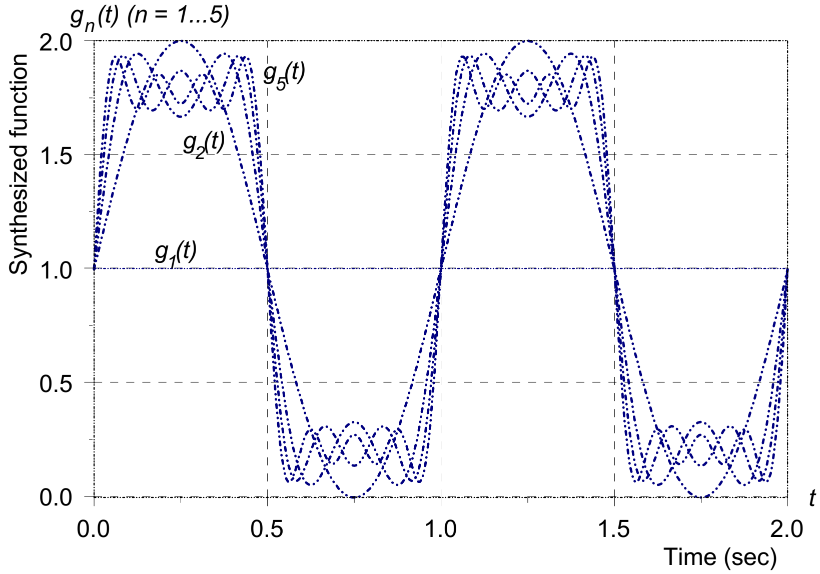

Figure 2: Synthesis of a periodic waveform by the summation of harmonically related components

The first term is a constant, and the four sinusoidal components are harmonically related, with a fundamental frequency of $\omega_0 = 2\pi$ radians/second and a fundamental period of $T = 2\pi/\omega_0 = 1$ second. (The constant term may be considered to be periodic with any arbitrary period, but is commonly considered to have a frequency of zero radians/second.) Figure 2 shows the evolution of the function that is formed as more of the individual terms are included into the summation. Notice that in all cases the period of the resulting $g_N(t)$ remains constant and equal to the period of the fundamental component (1 second). In this particular case it can be seen that the sum is tending toward a square wave.

The *Fourier series* [2,3] representation of a real periodic function $f(t)$ is based upon the summation of harmonically related sinusoidal components. If the period is T, then the harmonics are sinusoids with frequencies that are integer multiples of ω_0, that is the nth harmonic component has a frequency $n\omega_0 = 2\pi n/T$, and can be written

$$\begin{aligned} f_n(t) &= a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) && (7) \\ &= \mathcal{A}_n \sin(n\omega_0 t + \phi_n). && (8) \end{aligned}$$

In the first form the function $f_n(t)$ is written as a pair of sine and cosine functions with real coefficients a_n and b_n. The second form, in which the component is expressed as a single sinusoid with an amplitude $\mathcal{A}_n$ and a phase ϕ_n, is directly related to the first by the trigonometric relationship:

$$\mathcal{A}_n \sin(n\omega_0 t + \phi_n) = \mathcal{A}_n \sin\phi_n \cos(n\omega_0 t) + \mathcal{A}_n \cos\phi_n \sin(n\omega_0 t).$$

Equating coefficients,

$$\begin{aligned} a_n &= \mathcal{A}_n \sin\phi_n \\ b_n &= \mathcal{A}_n \cos\phi_n && (9) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_n &= \sqrt{a_n^2 + b_n^2} \\ \phi_n &= \tan^{-1}(a_n/b_n). \end{aligned} \tag{10}$$

The Fourier series representation of an arbitrary periodic waveform $f(t)$ (subject to some general conditions described later) is as an infinite sum of harmonically related sinusoidal components, commonly written in the following two equivalent forms

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)) \tag{11}$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\omega_0 t + \phi_n). \tag{12}$$

In either representation knowledge of the fundamental frequency ω_0, and the sets of Fourier coefficients $\{a_n\}$ and $\{b_n\}$ (or $\{\mathcal{A}_n\}$ and $\{\phi_n\}$) is sufficient to completely define the waveform $f(t)$.

A third, and completely equivalent, representation of the Fourier series expresses each of the harmonic components $f_n(t)$ in terms of complex exponentials instead of real sinusoids. The Euler formulas may be used to replace each sine and cosine terms in the components of Eq. (7) by a pair of complex exponentials

$$\begin{aligned} f_n(t) &= a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \\ &= \frac{a_n}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) + \frac{b_n}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) \\ &= \frac{1}{2}(a_n - jb_n)\, e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)\, e^{-jn\omega_0 t} \\ &= F_n e^{jn\omega_0 t} + F_{-n} e^{-jn\omega_0 t} \end{aligned} \tag{13}$$

where the new coefficients

$$\begin{aligned} F_n &= 1/2(a_n - jb_n) \\ F_{-n} &= 1/2(a_n + jb_n) \end{aligned} \tag{14}$$

are now complex numbers. With this substitution the Fourier series may be written in a compact form based upon harmonically related complex exponentials

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}. \tag{15}$$

This form of the series requires summation over all negative and positive values of n, where the coefficients of terms for positive and negative values of n are complex conjugates,

$$F_{-n} = \overline{F}_n, \tag{16}$$

so that knowledge of the coefficients F_n for $n \geq 0$ is sufficient to define the function $f(t)$.

Throughout thrse notes we adopt the nomenclature of using upper case letters to represent the Fourier coefficients in the *complex* series notation, so that the set of coefficients $\{G_n\}$ represent the function $g(t)$, and $\{Y_n\}$ are the coefficients of the function $y(t)$. The lower case coefficients $\{a_n\}$ and $\{b_n\}$ are used to represent the *real* Fourier coefficients of any function of time.

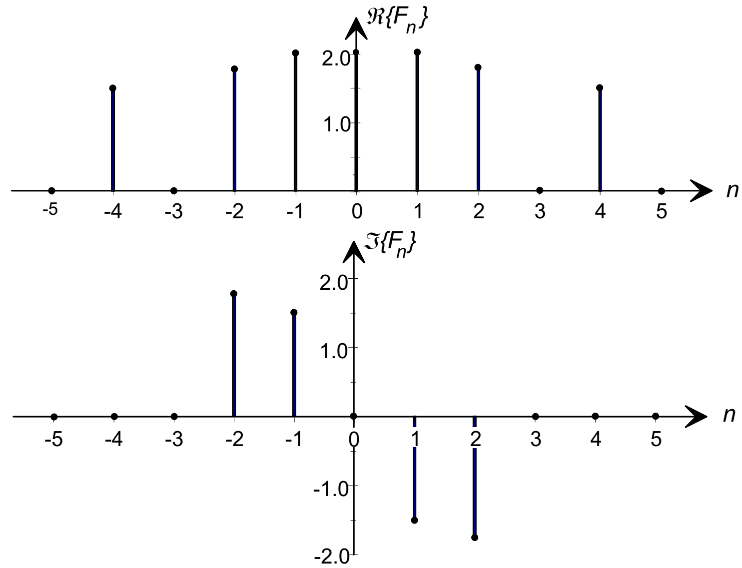

Figure 3: Spectral representation of the waveform discussed in Example 3.

■ Example

A periodic function $f(t)$ consists of five components

$$f(t) = 2 + 3\sin(100t) + 4\cos(100t) + 5\sin(200t + \pi/4) + 3\cos(400t).$$

It may be expressed as a finite complex Fourier series by expanding each term through the Euler formulas

$$\begin{aligned}
f(t) &= 2 + \frac{3}{2j}\left(e^{j100t} - e^{-j100t}\right) + \frac{4}{2}\left(e^{j100t} + e^{-j100t}\right) \\
&\quad + \frac{5}{2j}\left(e^{j(200t+\pi/4)} - e^{-j(200t+\pi/4)}\right) + \frac{3}{2}\left(e^{j400t} + e^{-j400t}\right) \\
&= 2 + \left(2 + \frac{3}{2j}\right)e^{j100t} + \left(2 - \frac{3}{2j}\right)e^{-j100t} \\
&\quad \left(\frac{5}{2\sqrt{2}} + \frac{5}{2\sqrt{2}j}\right)e^{j200t} + \left(\frac{5}{2\sqrt{2}} - \frac{5}{2\sqrt{2}j}\right)e^{-j200t} \\
&\quad + \frac{3}{2}e^{j400t} + \frac{3}{2}e^{-j400t}.
\end{aligned}$$

The fundamental frequency is $\omega_0 = 100$ radians/second, and the time-domain function contains harmonics $n = 1, 2, 3$, and 4. The complex Fourier coefficients are

$$\begin{array}{ll}
F_0 = 2 & \\
F_1 = 2 - \frac{3}{2}j & F_{-1} = 2 + \frac{3}{2}j \\
F_2 = \frac{5}{2\sqrt{2}}(1 - 1j) & F_{-2} = \frac{5}{2\sqrt{2}}(1 + 1j) \\
F_3 = 0 & F_{-3} = 0 \\
F_4 = \frac{3}{2} & F_{-4} = \frac{3}{2}
\end{array}$$

The finite Fourier series may be written in the complex form using these coefficients as

$$f(t) = \sum_{n=-5}^{5} F_n e^{jn100t}$$

and plotted with real and imaginary parts as in Fig. 3.

The values of the Fourier coefficients, in any of the three above forms, are effectively measures of the amplitude and phase of the harmonic component at a frequency of $n\omega_0$. The *spectrum* of a periodic waveform is the set of all of the Fourier coefficients, for example $\{\mathcal{A}_n\}$ and $\{\phi_n\}$, expressed as a function of frequency. Because the harmonic components exist at discrete frequencies, periodic functions are said to exhibit *line spectra*, and it is common to express the spectrum graphically with frequency ω as the independent axis, and with the Fourier coefficients plotted as lines at intervals of ω_0. The first two forms of the Fourier series, based upon Eqs. (7) and (8), generate "one-sided" spectra because they are defined from positive values of n only, whereas the complex form defined by Eq. (15) generates a "two-sided" spectrum because its summation requires positive and negative values of n. Figure 3 shows the complex spectrum for the finite series discussed in Example 2.

2.1 Computation of the Fourier Coefficients

The derivation of the expressions for computing the coefficients in a Fourier series is beyond the scope of this book, and we simply state without proof that if $f(t)$ is periodic with period T and fundamental frequency ω_0, in the complex exponential form the coefficients F_n may be computed from the equation

$$F_n = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)e^{-jn\omega_0 t}dt \tag{17}$$

where the initial time t_1 for the integration is arbitrary. The integral may be evaluated over any interval that is one period T in duration.

The corresponding formulas for the sinusoidal forms of the series may be derived directly from Eq. (17). From Eq. (14) it can be seen that

$$\begin{aligned} a_n &= F_n + F_{-n} \\ &= \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right] dt \\ &= \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos(n\omega_0 t)dt \end{aligned} \tag{18}$$

and similarly

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin(n\omega_0 t)dt \tag{19}$$

The calculation of the coefficients for a given periodic time function $f(t)$ is known as Fourier *analysis* or *decomposition* because it implies that the waveform can be "decomposed" into its spectral components. On the other hand, the expressions that express $f(t)$ as a Fourier series summation (Eqs. (11), (12), and (15)) are termed Fourier *synthesis* equations because the imply that $f(t)$ could be created (synthesized) from an infinite set of harmonically related oscillators.

Table 1 summarizes the analysis and synthesis equations for the sinusoidal and complex exponential formulations of the Fourier series.

	Sinusoidal formulation	Exponential formulation
Synthesis:	$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$	$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}$
Analysis:	$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos(n\omega_0 t)dt$ $b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin(n\omega_0 t)dt$	$F_n = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)e^{-jn\omega_0 t}dt$

Table 1: Summary of analysis and synthesis equations for Fourier analysis and synthesis.

■ Example

Find the complex and real Fourier series representations of the periodic square wave $f(t)$ with period T,

$$f(t) = \begin{cases} 1 & 0 \le t < T/2, \\ 0 & T/2 \le t < T \end{cases}$$

as shown in Fig. 4.

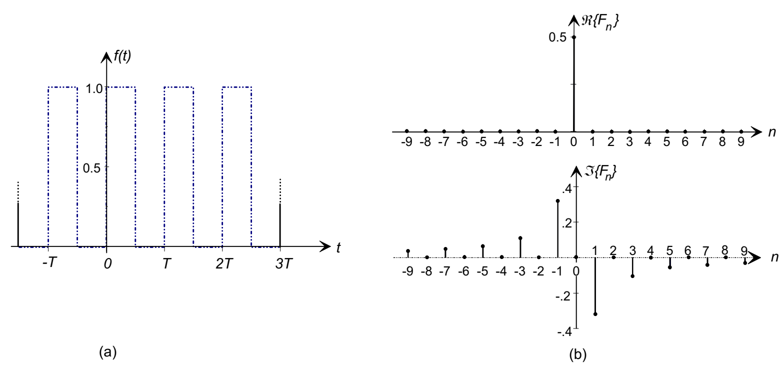

Figure 4: A periodic square wave and its spectrum.

Solution: The complex Fourier series is defined by the synthesis equation

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}. \tag{20}$$

In this case the function is non-zero for only half of the period, and the integration limits can be restricted to this range. The zero frequency coefficient F_0 must be computed separately:

$$F_0 = \frac{1}{T}\int_0^{T/2} e^{j0}dt = \frac{1}{2}, \tag{21}$$

and all of the other coefficients are:

$$\begin{aligned} F_n &= \frac{1}{T}\int_0^{T/2}(1)e^{-jn\omega_0 t}dt \\ &= \frac{-1}{jn\omega_0 T}\left[e^{-jn\omega_0 t}\Big|_0^{T/2}\right. \\ &= \frac{j}{2n\pi}\left[e^{-jn\pi}-1\right] \end{aligned} \qquad (22)$$

since $\omega_0 T = 2\pi$. Because $e^{-jn\pi} = -1$ when n is odd and $+1$ when n is even,

$$F_n = \begin{cases} -j/n\pi, & \text{if n is odd;} \\ 0, & \text{if n is even.} \end{cases}$$

The square wave can then be written as the complex Fourier series

$$f(t) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{j}{(2m-1)\pi}\left(e^{-j(2m-1)\omega_0 t} - e^{j(2m-1)\omega_0 t}\right). \qquad (23)$$

where the terms $F_n e^{j\omega_n t}$ and $F_{-n}e^{-j\omega_n t}$ have been combined in the summation.

If the Euler formulas are be used to expand the complex exponentials, the cosine terms cancel, and the resulting series involves only sine terms:

$$\begin{aligned} f(t) &= \frac{1}{2} + \sum_{m=1}^{\infty}\frac{2}{(2m-1)\pi}\sin\left((2m-1)\,\omega_0 t\right) \qquad (24) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t) + \dots\right). \end{aligned}$$

Comparison of the terms in this series with the components of the waveform synthesized in Example 2, and shown in Fig. 2, shows how a square wave may be progressively approximated by a finite series.

2.2 Properties of the Fourier Series

A full discussion of the properties of the Fourier series is beyond the scope of this book, and the interested reader is referred to the references. Some of the more important properties are summarized below.

(1) **Existence of the Fourier Series** For the series to exist, the integral of Eq. (17) must converge. A set of three sufficient conditions, known as the Dirichelet conditions, guarantee the existence of a Fourier series for a given periodic waveform $f(t)$. They are

- The function f(t) must be *absolutely integrable* over any period, that is

$$\int_{t_1}^{t_1+T} |f(t)|\, dt < \infty \qquad (25)$$

for any t_1.

- There must be at most a finite number of maxima and minima in the function $f(t)$ within any period.
- There must be at most a finite number of discontinuities in the function $f(t)$ within any period, and all such discontinuities must be finite in magnitude.

These requirements are satisfied by almost all waveforms found in engineering practice. The Dirichelet conditions are a *sufficient* set of conditions to guarantee the existence of a Fourier series representation. They are not *necessary* conditions, and there are some functions that have a Fourier series representation without satisfying all three conditions.

(2) **Linearity of the Fourier Series Representation** The Fourier analysis and synthesis operations are linear. Consider two periodic functions $g(t)$ and $h(t)$ with identical periods T, and their complex Fourier coefficients

$$\begin{aligned} G_n &= \frac{1}{T}\int_0^T g(t)e^{-jn\omega_0 t}dt \\ H_n &= \frac{1}{T}\int_0^T h(t)e^{-jn\omega_0 t}dt \end{aligned}$$

and a third function defined as a weighted sum of $g(t)$ and $h(t)$

$$f(t) = ag(t) + bh(t)$$

where a and b are constants. The linearity property, which may be shown by direct substitution into the integral, states that the Fourier coefficients of $f(t)$ are

$$F_n = aG_n + bH_n,$$

that is the Fourier series of a weighted sum of two time-domain functions is the weighted sum of the individual series.

(3) **Even and Odd Functions** If $f(t)$ exhibits symmetry about the $t = 0$ axis the Fourier series representation may be simplified. If $f(t)$ is an even function of time, that is $f(-t) = f(t)$, the complex Fourier series has coefficients F_n that are purely real, with the result that the real series contains only cosine terms, so that Eq. (11) simplifies to

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t). \tag{26}$$

Similarly if $f(t)$ is an odd function of time, that is $f(-t) = -f(t)$, the coefficients F_n are imaginary, and the one-sided series consists of only sine terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t). \tag{27}$$

Notice that an odd function requires that $f(t)$ have a zero average value.

(4) **The Fourier Series of a Time Shifted Function** If the periodic function $f(t)$ has a Fourier series with complex coefficients F_n, the series representing a "time-shifted" version $g(t) = f(t+\tau)$ has coefficients $e^{-jn\omega_0\tau}F_n$. If

$$F_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt$$

then

$$G_n = \frac{1}{T}\int_0^T f(t+\tau)e^{-jn\omega_0 t}dt.$$

Changing the variable of integration $\nu = t+\tau$ gives

$$\begin{aligned} G_n &= \frac{1}{T}\int_\tau^{\tau+T} f(\nu)e^{-jn\omega_0(\nu-\tau)}d\nu \\ &= e^{jn\omega_0\tau}\frac{1}{T}\int_\tau^{\tau+T} f(\nu)e^{-jn\omega_0\nu t}d\nu \\ &= e^{jn\omega_0\tau}F_n. \end{aligned}$$

If the nth spectral component is written in terms of its magnitude and phase

$$f_n(t) = \mathcal{A}_n \sin(n\omega_0 t + \phi_n)$$

then

$$\begin{aligned} f_n(t+\tau) &= \mathcal{A}_n \sin\left(n\omega_0(t+\tau)+\phi_n\right) \\ &= \mathcal{A}_n \sin\left(n\omega_0 t + \phi_n + n\omega_0\tau\right). \end{aligned}$$

The additional phase shift $n\omega_0\tau$, caused by the time shift τ, is directly proportional to the frequency of the component $n\omega_0$.

(5) <u>**Interpretation of the Zero Frequency Term**</u> The coefficients F_0 in the complex series and a_0 in the real series are somewhat different from all of the other terms for they correspond to a harmonic component with zero frequency. The complex analysis equation shows that

$$F_0 = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$$

and the real analysis equation gives

$$\frac{1}{2}a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$$

which are both simply the average value of the function over one complete period. If a function $f(t)$ is modified by adding a constant value to it, the only change in its series representation is in the coefficient of the zero-frequency term, either F_0 or a_0.

■ Example

Find a Fourier series representation for the periodic "saw-tooth" waveform with period T

$$f(t) = \frac{2}{T}t, \qquad |t| < T/2$$

shown in Fig. 5.

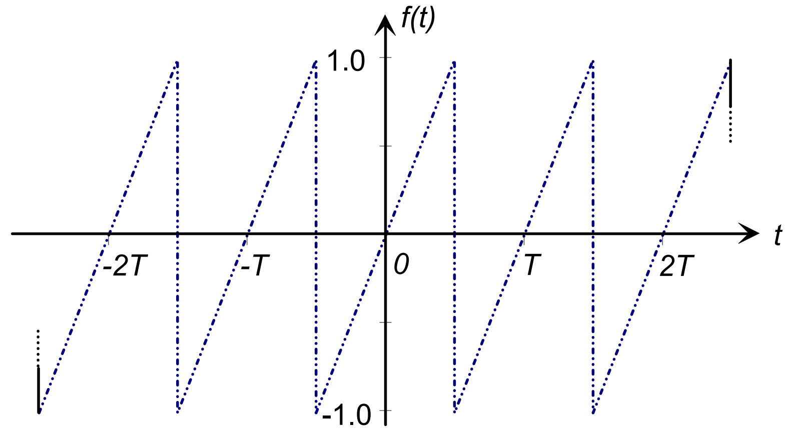

Figure 5: A periodic saw-tooth waveform.

Solution: The complex Fourier coefficients are

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} \frac{2}{T} t e^{jn\omega_0 t} dt, \tag{28}$$

and integrating by parts

$$\begin{aligned} F_n &= \frac{2j}{n\omega_0 T^2}\left[t e^{-jn\omega_0 t}\Big|_{-T/2}^{T/2} + \int_{-T/2}^{T/2} \frac{1}{jn\omega_0} e^{-jn\omega_0 t} dt \right. \\ &= \frac{j}{2n\pi}\left[e^{-jn\pi} + e^{jn\pi}\right] + 0 \\ &= \frac{j}{n\pi}\cos(n\pi) \\ &= \frac{j(-1)^n}{n\pi} \qquad\qquad n \neq 0, \end{aligned} \tag{29}$$

since $\cos(n\pi) = (-1)^n$. The zero frequency coefficient must be evaluated separately:

$$F_0 = \frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{2}{T}t\right) dt = 0. \tag{30}$$

The Fourier series is:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{j(-1)^n}{n\pi}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\omega_0 t) nonumber \qquad (31) \\ &= \frac{2}{\pi}\left(\sin(\omega_0 t) - \frac{1}{2}\sin(2\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) - \frac{1}{4}\sin(4\omega_0 t) + \ldots\right). \qquad (32) \end{aligned}$$

■ **Example**

Find a Fourier series representation of the function

$$f(t) = \begin{cases} \frac{1}{4} + \frac{2}{T}t & -5T/8 \le t < -T/8, \\ \\ \frac{5}{4} + \frac{2}{T}t & -T/8 \le t < 3T/8 \end{cases}$$

as shown in Fig. 6.

Solution: If $f(t)$ is rewritten as:

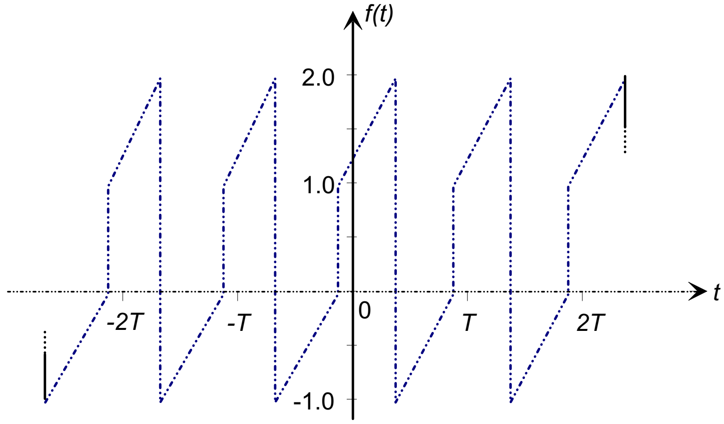

Figure 6: A modified saw-tooth function having a shift in amplitude and time origin

$$f(t) = \begin{cases} 0 + \frac{2}{T}(t + T/8) & -5T/8 \le t < -T/8, \\ \\ 1 + \frac{2}{T}(t + T/8) & -T/8 \le t < 3T/8, \end{cases} \tag{33}$$

then

$$f(t) = f_1(t + T/8) + f_2(t + T/8), \tag{34}$$

where $f_1(t)$ is the square wave function analysed in Example 2.1:

$$f(t) = \begin{cases} 1 & 0 \le t < T/2, \\ 0 & T/2 \le t < T, \end{cases} \tag{35}$$

and $f_2(t)$ is the saw-tooth function of Example 2.2.

$$f(t) = \frac{2}{T}t, \qquad |t| < T/2. \tag{36}$$

Therefore the function $f(t)$ is a *time shifted* version of the *sum* of two functions for which we already know the Fourier series. The Fourier series for $f(t)$ is the sum (Linearity Property) of phase shifted versions (Time Shifting Property) of the pair of Fourier series derived in Examples 2.1 and 2.2. The time shift of $T/8$ seconds adds a phase shift of $n\omega_0 T/8 = n\pi/4$ radians to each component

$$\begin{aligned} f_1(t+T/8) &= \frac{1}{2} + \frac{2}{\pi}\left(\sin(\omega_0 t + \pi/4) + \frac{1}{3}\sin(3\omega_0 t + 3\pi/4)\right. \\ &\quad \left. + \frac{1}{5}\sin(5\omega_0 t + 5\pi/4) + \frac{1}{7}\sin(7\omega_0 t + 7\pi/4) + \ldots\right) \quad (37) \\ f_2(t+T/8) &= \frac{2}{\pi}\left(\sin(\omega_0 t + \pi/4) - \frac{1}{2}\sin(2\omega_0 t + \pi/2) + \frac{1}{3}\sin(3\omega_0 t + 3\pi/4)\right. \\ &\quad \left. - \frac{1}{4}\sin(4\omega_0 t + \pi) + \frac{1}{5}\sin(5\omega_0 t + 5\pi/4) - \ldots\right). \quad (38) \end{aligned}$$

The sum of these two series is

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(2\sin(\omega_0 t + \pi/4) - \frac{1}{2}\sin(2\omega_0 t + \pi/2) + \frac{2}{3}\sin(3\omega_0 t + 3\pi/4)\right. \quad (39) \\ &\quad \left. - \frac{1}{4}\sin(4\omega_0 t + \pi) + \frac{2}{5}\sin(5\omega_0 t + 5\pi/4) - \frac{1}{6}\sin(6\omega_0 t + 3\pi/2) + \ldots\right). \quad (40) \end{aligned}$$

---

3 The Response of Linear Systems to Periodic Inputs

Consider a linear single-input, single-output system with a frequency response function $H(j\omega)$. Let the input $u(t)$ be a periodic function with period T, and assume that all initial condition transient components in the output have decayed to zero. Because the input is a periodic function it can be written in terms of its complex or real Fourier series

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} U_n e^{jn\omega_0 t} \quad (41) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\omega_0 t + \phi_n) \quad (42) \end{aligned}$$

The nth real harmonic input component, $u_n(t) = \mathcal{A}_n \sin(n\omega_0 t + \phi_n)$, generates an output sinusoidal component $y_n(t)$ with a magnitude and a phase that is determined by the system's frequency response function $H(j\omega)$:

$$y_n(t) = |H(jn\omega_0)|\, \mathcal{A}_n \sin(n\omega_0 t + \phi_n + \angle H(jn\omega_0)). \quad (43)$$

The principle of superposition states that the total output $y(t)$ is the sum of all such component outputs, or

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t) \\ &= \frac{1}{2}a_0 H(j0) + \sum_{n=1}^{\infty} \mathcal{A}_n\, |H(jn\omega_0)| \sin\left(n\omega_0 t + \phi_n + \angle H(jn\omega_0)\right), \quad (44) \end{aligned}$$

which is itself a Fourier series with the same fundamental and harmonic frequencies as the input. The output $y(t)$ is therefore also a periodic function with the same period T as the input, but because the system frequency response function has modified the relative magnitudes and the phases of the components, the waveform of the output $y(t)$ differs in form and appearance from the input $u(t)$.

In the complex formulation the input waveform is decomposed into a set of complex exponentials $u_n(t) = U_n e^{jn\omega_0 t}$. Each such component is modified by the system frequency response so that the output component is

$$y_n(t) = H(jn\omega_0)U_n e^{jn\omega_0 t} \tag{45}$$

and the complete output Fourier series is

$$y(t) = \sum_{n=-\infty}^{\infty} y_n(t) = \sum_{n=-\infty}^{\infty} H(jn\omega_0)U_n e^{jn\omega_0 t}. \tag{46}$$

■ Example

The first order electrical network shown in Fig. 7 is excited with the saw-tooth function discussed in Example 2.2. Find an expression for the series representing the output

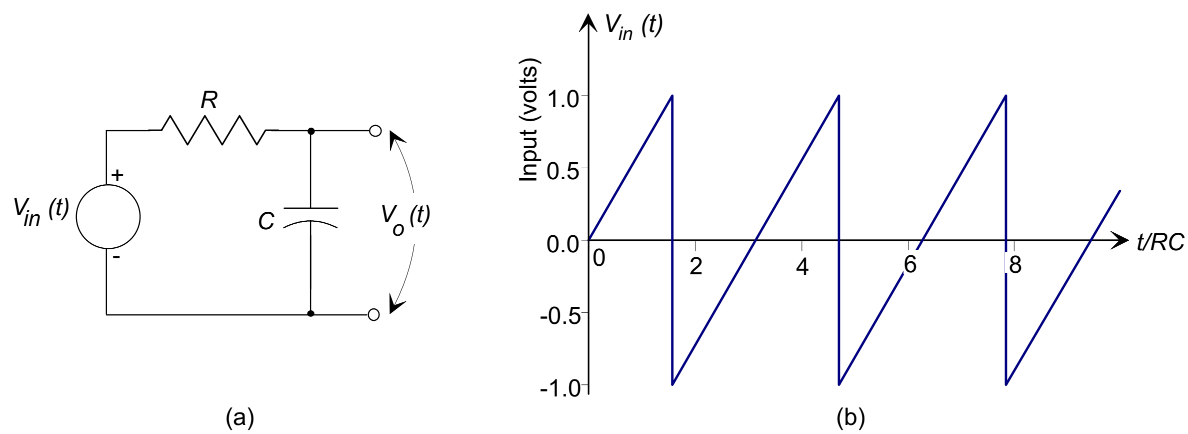

Figure 7: A first-order electrical system (a), driven by a saw-tooth input waveform (b).

$V_o(t)$.

Solution: The electrical network has a transfer function

$$H(s) = \frac{1}{RCs + 1}, \tag{47}$$

and therefore has a frequency response function

$$|H(j\omega)| = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \tag{48}$$

$$\angle H(j\omega) = \tan^{-1}(-\omega RC). \tag{49}$$

From Example 2.2, the input function $u(t)$ may be represented by the Fourier series

$$u(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\omega_0 t). \tag{50}$$

At the output the series representation is

$$y(t) = \sum_{n=1}^{\infty} |H(jn\omega_0)| \frac{2(-1)^{n+1}}{n\pi} \sin(n\omega_0 t + \angle H(jn\omega_0))$$
$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\sqrt{(n\omega_0 RC)^2+1}} \sin\left(n\omega_0 t + \tan^{-1}(-n\omega_0 RC)\right). \quad (51)$$

As an example consider the response if the period of the input is chosen to be $T = \pi RC$, so that $\omega_0 = 2/(RC)$, then

$$y(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\sqrt{(2n)^2+1}} \sin\left(\frac{2n}{RC}t + \tan^{-1}(-2n)\right).$$

Figure 8 shows the computed response, found by summing the first 100 terms in the Fourier Series.

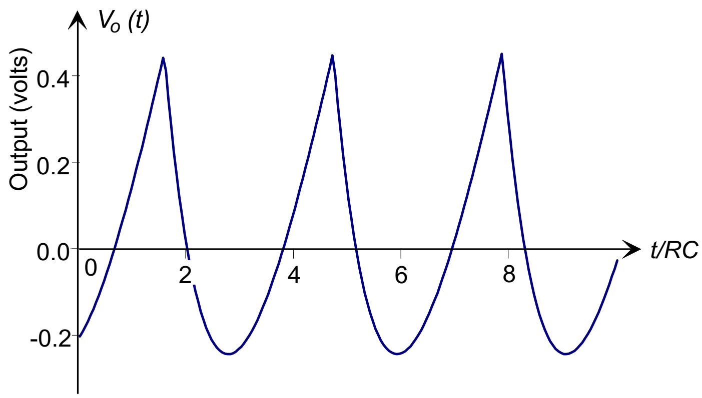

Figure 8: Response of first-order electrical system to a saw-tooth input

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Equations (46) and (45) show that the output component Fourier coefficients are products of the input component coefficient and the frequency response evaluated at the frequency of the harmonic component. No new frequency components are introduced into the output, but the form of the output $y(t)$ is modified by the redistribution of the input component amplitudes and phase angles by the frequency response $H(jn\omega_0)$. If the system frequency response exhibits a low-pass characteristic with a cut-off frequency within the spectrum of the input $u(t)$, the high frequency components are attenuated in the output, with a resultant general "rounding" of any discontinuities in the input. Similarly a system with a high-pass characteristic emphasizes any high frequency component in the input. A system having lightly damped complex conjugate pole pairs exhibits resonance in its response at frequencies close to the undamped natural frequency of the pole pair. It is entirely possible for a periodic function to excite this resonance through one of its harmonics even though the fundamental frequency is well removed from the resonant frequency as is shown in Example 3.

■ **Example**

A cart, shown in Fig. 9a, with mass $m = 1.0$ kg is supported on low friction bearings that exhibit a viscous drag $B = 0.2$ N-s/m, and is coupled through a spring with stiffness K = 25 N/m to a velocity source with a magnitude of 10 m/s, but which switches direction every π seconds as shown in Fig. 9b.

$$V_{in}(t) = \begin{cases} 10 \text{ m/sec} & 0 \le t < \pi, \\ -10 \text{ m/sec} & \pi \le t < 2\pi \end{cases}$$

The task is to find the resulting velocity of the mass $v_m(t)$.

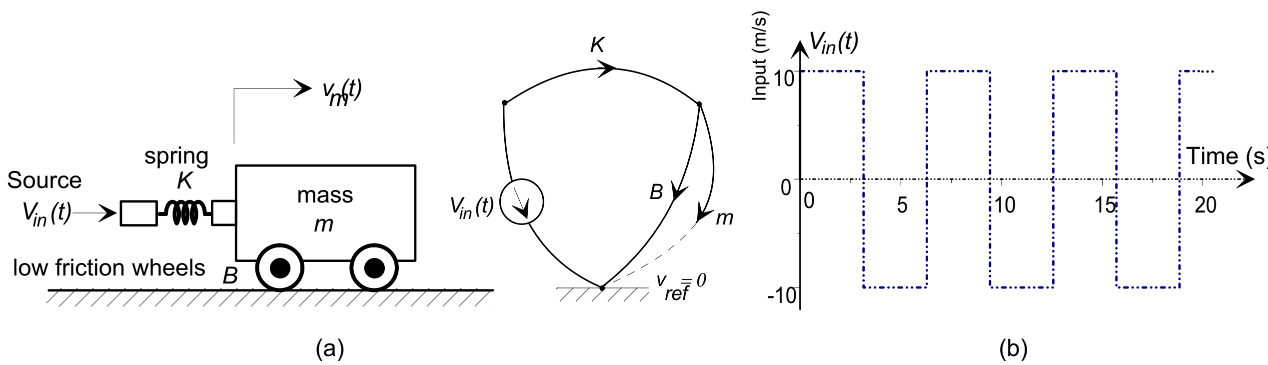

Figure 9: A second-order system and its linear graph, together with its input waveform $V_{in}(t)$.

Solution: The system has a transfer function

$$H(s) = \frac{K/m}{s^2 + (B/m)\,s + K/m} \tag{52}$$

$$= \frac{25}{s^2 + 0.2s + 25} \tag{53}$$

and an undamped natural frequency $\omega_n = 5$ rad/s and a damping ratio $\zeta = 0.02$. It is therefore lightly damped and has a strong resonance in the vicinity of 5 rad/s.

The input $\Omega(t)$ has a period of $T = 2\pi$ s, and fundamental frequency of $\omega_0 = 2\pi/T = 1$ rad/s. The Fourier series for the input may be written directly from Example 2.1, and contains only odd harmonics:

$$u(t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)\omega_0 t\right) \tag{54}$$

$$= \frac{20}{\pi}\left(\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \ldots\right). \tag{55}$$

From Eq. (ii) the frequency response of the system is

$$H(j\omega) = \frac{25}{(25 - \omega^2) + j0.2\omega}, \tag{56}$$

which when evaluated at the harmonic frequencies of the input $n\omega_0 = n$ radians/sec. is

$$H(jn\omega_0) = \frac{25}{(25 - n^2) + j0.2n}. \tag{57}$$

The following table summarizes the the first five odd spectral components at the system input and output. Fig. 10a shows the computed frequency response magnitude for the system and the relative gains and phase shifts (rad) associated with the first five terms in the series.

$n\omega_0$	u_n	$\lvert H(jn\omega_0)\rvert$	$\angle H(jn\omega_0)$	y_n
1	$6.366\sin(t)$	1.041	-0.008	$6.631\sin(t-0.008)$
3	$2.122\sin(3t)$	1.561	-0.038	$3.313\sin(t-0.038)$
5	$1.273\sin(5t)$	25.00	-1.571	$31.83\sin(t-1.571)$
7	$0.909\sin(7t)$	1.039	-3.083	$0.945\sin(t-3.083)$
9	$0.707\sin(9t)$	0.446	-3.109	$0.315\sin(t-3.109)$

The resonance in $|H(j\omega)|$ at the undamped natural frequency $\omega_n = 5$ rad/s has a large effect on the relative amplitude of the 5th harmonic in the output $y(t)$. Figure 10b shows the system input and output waveforms. The effect of the resonance can be clearly seen, for the output appears to be almost sinusoidal at a frequency of 5 rad/s. In fact the output is still a periodic waveform with a period of 2π seconds but the fifth harmonic component dominates the response and makes it appear to be sinusoidal at its own frequency.

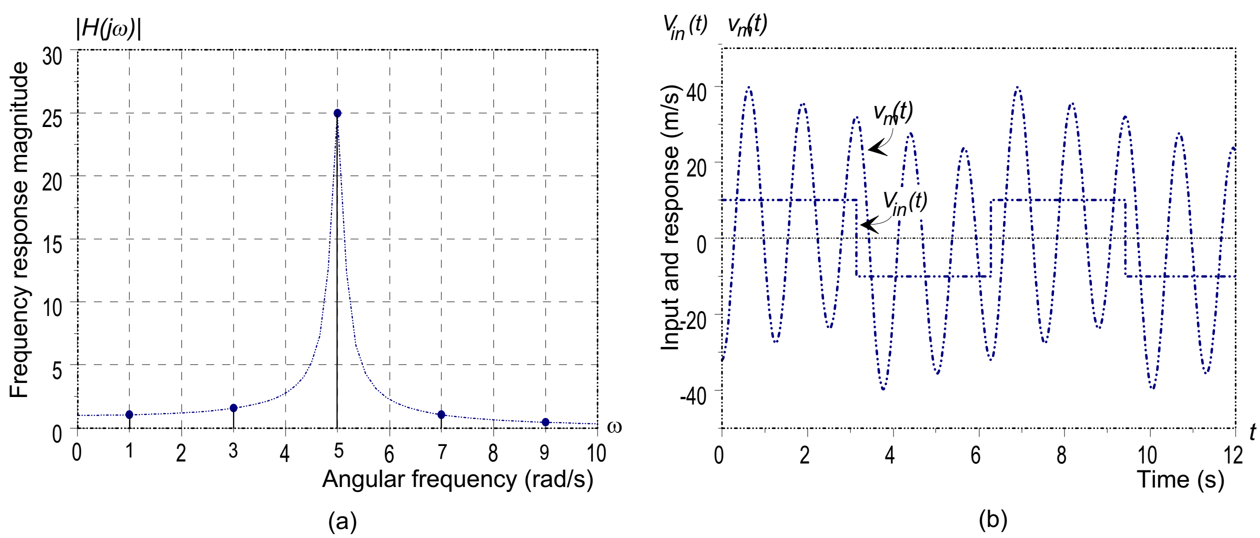

Figure 10: (a) The frequency response magnitude function of the mechanical system in Example 8, and (b) an input square wave function and its response.

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4 Fourier Analysis of Transient Waveforms

Many waveforms found in practice are not periodic and therefore cannot be analyzed directly using Fourier series methods. A large class of system excitation functions can be characterized as *aperiodic*, or transient, in nature. These functions are limited in time, they occur only once, and decay to zero as time becomes large [2,4,5].

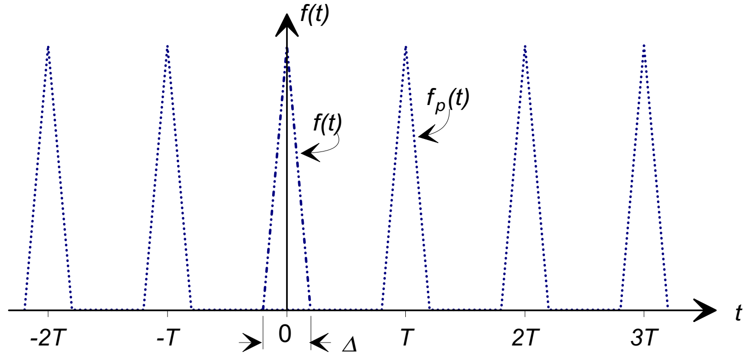

Figure 11: Periodic extension of a transient waveform

Consider a function $f(t)$ of duration Δ that exists only within a defined interval $t_1 < t \leq t_1 + \Delta$, and is identically zero outside of this interval. We begin by making a simple assumption; namely that in observing the transient phenomenon $f(t)$ within any finite interval that encompasses it, we have observed a fraction of a single period of a periodic function with a very large period; much larger than the observation interval. Although we do not know what the duration of this hypothetical period is, it is assumed that $f(t)$ will repeat itself at some time in the distant future, but in the meantime it is assumed that this periodic function remains identically zero for the rest of its period outside the observation interval.

The analysis thus conjectures a new function $f_p(t)$, known as a *periodic extension* of $f(t)$, that repeats every T seconds ($T > \Delta$), but at our discretion we can let T become very large. Figure 11 shows the hypothetical periodic extension $f_p(t)$ created from the observed $f(t)$. As observers of the function $f_p(t)$ we need not be concerned with its pseudo-periodicity because we will never be given the opportunity to experience it outside the first period, and furthermore we can assume that if $f_p(t)$ is the input to a linear system, T is so large that the system response decays to zero before the arrival of the second period. Therefore we assume that the response of the system to $f(t)$ and $f_p(t)$ is identical within our chosen observation interval. The important difference between the two functions is that $f_p(t)$ is periodic, and therefore has a Fourier series description.

The development of Fourier analysis methods for transient phenomena is based on the limiting behavior of the Fourier series describing $f_p(t)$ as the period T approaches infinity. Consider the behavior of the Fourier series of a simple periodic function as its period T is varied; for example an even periodic pulse function $f(t)$ of fixed width Δ:

$$f(t) = \begin{cases} 1 & |t| < \Delta/2 \\ 0 & \Delta/2 \leq |t| \leq T - \Delta/2 \end{cases} \tag{58}$$

——————- as shown in Figure 12. Assume that the pulse width Δ remains constant as the period T varies. The Fourier coefficients in complex form are

$$\begin{aligned} F_n &= \frac{1}{T}\int_{-\Delta/2}^{\Delta/2} e^{-jn\omega_0 t} dt \\ &= \frac{j}{2n\pi}\left[e^{-jn\omega_0\Delta/2} - e^{jn\omega_0\Delta/2}\right] \\ &= \frac{1}{n\pi}\sin\left(n\pi\Delta/T\right) \\ &= \frac{\Delta}{T}\frac{\sin\left(n\pi\Delta/T\right)}{\left(n\pi\Delta/T\right)} \qquad n \neq 0 \end{aligned} \tag{59}$$

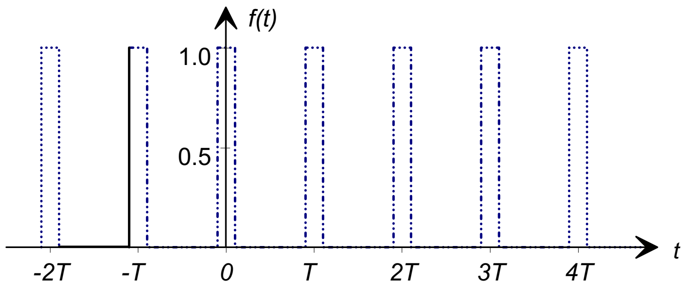

Figure 12: A periodic rectangular pulse function of fixed duration Δ but varying period T.

and

$$F_0 = \frac{1}{T}\int_{-\Delta/2}^{\Delta/2} 1dt = \frac{\Delta}{T} \tag{60}$$

and the spectral lines are spaced along the frequency axis at intervals of $\omega_0 = 2\pi/T$ rad/s. To investigate the behavior of the spectrum as the period T is altered, we define a continuous function of frequency

$$F(\omega) = \frac{\sin(\omega\Delta/2)}{(\omega\Delta/2)}$$

and note that the Fourier coefficients may be computed directly from $F(\omega)$

$$F_n = \frac{\Delta}{T}\left.\frac{\sin(\omega\Delta/2)}{(\omega\Delta/2)}\right|_{\omega=2\pi n/T} \tag{61}$$

$$= \frac{\Delta}{T}\left.F(\omega)\right|_{\omega=n\omega_0} \tag{62}$$

The function $F(\omega)$ depends only on the pulse width Δ, and is independent of the period T. As T is changed, apart from the amplitude scaling factor Δ/T, the frequency dependence of the Fourier coefficients is defined by $F(\omega)$; the relative strength of the nth complex harmonic component, at a frequency $n\omega_0$, is defined by $F(n\omega_0)$. The function $F(\omega)$ is therefore an *envelope* function that depends only on $f(t)$ and not on the length of the assumed period. Figure 13 shows examples of the line spectra for the periodic pulse train as the period T is changed. The following general observations on the behavior of the Fourier coefficients as T varies can be made:

(1) As the repetition period T increases, the fundamental frequency ω_0 decreases, and the spacing between adjacent lines in the spectrum decreases.

(2) As the repetition period T increases, the scaling factor Δ/T decreases, causing the magnitude of all of the spectral lines to be diminished. In the limit as T approaches infinity, the amplitude of the individual lines becomes infinitesimal.

(3) The "shape" of the spectrum is defined by the function $F(\omega)$ and is independent of T.

Assume that we have an aperiodic function $f(t)$ that is non-zero only for a defined time interval Δ, and without loss of generality assume that the interval is centered around the time origin ($t = 0$). Then assume a periodic extension $f_p(t)$ of $f(t)$ with period T that fully encompasses the interval Δ. The Fourier series description of $f_p(t)$ is contained in the analysis and synthesis equations

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_p(t)e^{-jn\omega_0 t}dt \tag{63}$$

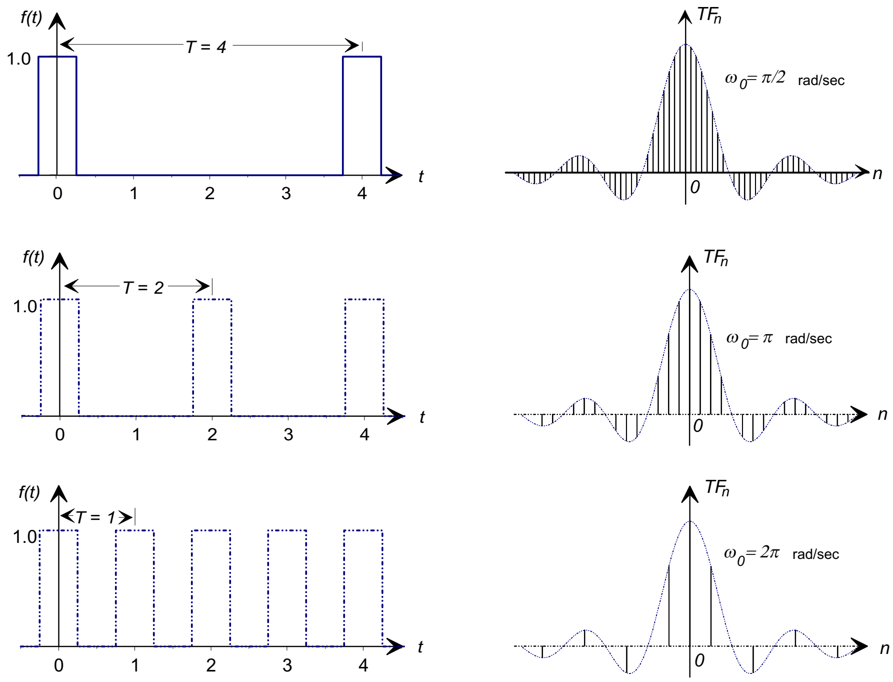

Figure 13: Line spectra of periodic extensions of an even rectangular pulse function.

$$f_p(t) \;=\; \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}. \tag{64}$$

These two equations may be combined by substituting for F_n in the synthesis equation,

$$f_p(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{\omega_0}{2\pi} \int_{-T/2}^{T/2} f_p(t) e^{-jn\omega_0 t} dt \right\} e^{jn\omega_0 t} \tag{65}$$

where the substitution $\omega_0/2\pi = 1/T$ has also been made.

The period T is now allowed to become arbitrarily large, with the result that the fundamental frequency ω_0 becomes very small and we write $\omega_0 = \delta\omega$. We define $f(t)$ as the limiting case of $f_p(t)$ as T approaches infinity, that is

$$\begin{aligned} f(t) &= \lim_{T\to\infty} f_p(t) \\ &= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-T/2}^{T/2} f_p(t) e^{-jn\delta\omega t} dt \right\} e^{jn\delta\omega t} \delta\omega \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-\infty}^{-\infty} f(t) e^{-j\omega t} dt \right\} e^{j\omega t} d\omega \end{aligned} \tag{66}$$

where in the limit the summation has been replaced by an integral. If the function inside the braces is defined to be $F(j\omega)$, Equation 66 may be expanded into a pair of equations, known as the *Fourier transform pair*:

$$F(j\omega) \;=\; \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{67}$$

$$f(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \tag{68}$$

which are the equations we seek.

Equation 67 is known as the *forward* Fourier transform, and is analogous to the analysis equation of the Fourier series representation. It expresses the time-domain function $f(t)$ as a function of frequency, but unlike the Fourier series representation it is a continuous function of frequency. Whereas the Fourier series coefficients have units of amplitude, for example volts or Newtons, the function $F(j\omega)$ has units of amplitude density, that is the total "amplitude" contained within a small increment of frequency is $F(j\omega)\delta\omega/2\pi$.

Equation 68 defines the *inverse* Fourier transform. It allows the computation of the time-domain function from the frequency domain representation $F(j\omega)$, and is therefore analogous to the Fourier series synthesis equation. Each of the two functions $f(t)$ or $F(j\omega)$ is a complete description of the function and Equations 67 and 68 allow the transformation between the domains.

We adopt the convention of using lower-case letters to designate time-domain functions, and the same upper-case letter to designate the frequency-domain function. We also adopt the nomenclature

$$f(t) \overset{F.T.}{\Longleftrightarrow} F(j\omega)$$

as denoting the bidirectional Fourier transform relationship between the time and frequency-domain representations, and we also frequently write

$$\begin{aligned} F(j\omega) &= \mathcal{F}\{f(t)\} \\ f(t) &= \mathcal{F}^{-1}\{F(j\omega)\} \end{aligned}$$

as denoting the operation of taking the forward $\mathcal{F}\{\}$, and inverse $\mathcal{F}^{-1}\{\}$ Fourier transforms respectively.

4.1 Fourier Transform Examples

In this section we present five illustrative examples of Fourier transforms of common time domain functions.

■ Example

Find the Fourier transform of the pulse function

$$f(t) = \begin{cases} a & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$

shown in Figure 14.

Solution: From the definition of the forward Fourier transform

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt & (69) \\ &= a\int_{-T/2}^{T/2} e^{-j\omega t}dt & (70) \\ &= a\left[\frac{j}{\omega}e^{-j\omega t}\right|_{T/2}^{T/2} & (71) \\ &= \frac{ja}{\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right] & (72) \\ &= aT\frac{\sin(\omega T/2)}{\omega T/2}. & (73) \end{aligned}$$

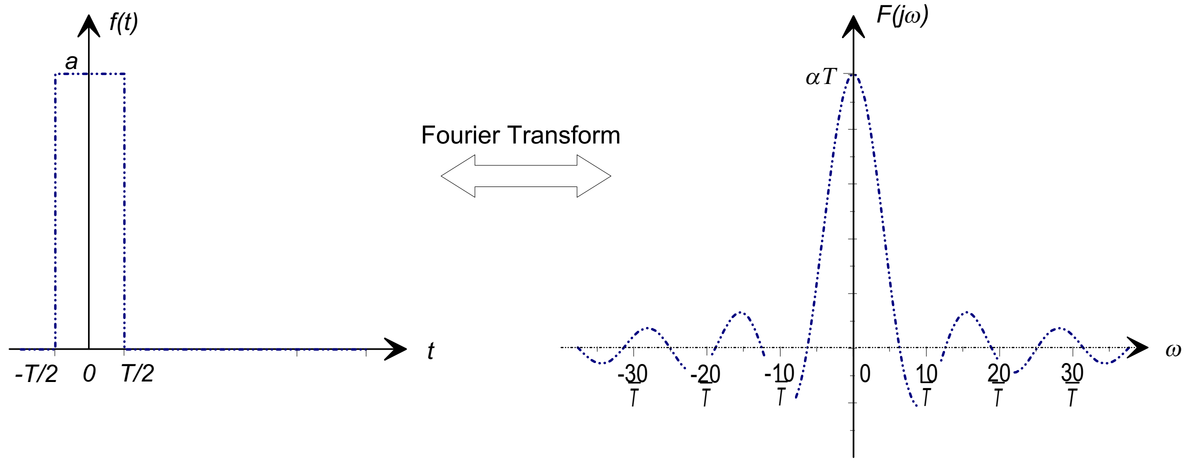

Figure 14: An even aperiodic pulse function and its Fourier transform

The Fourier transform of the rectangular pulse is a real function, of the form $(\sin x)/x$ centered around the $j\omega = 0$ axis. Because the function is real, it is sufficient to plot a single graph showing only $|F(j\omega)|$ as in Figure 14. Notice that while $F(j\omega)$ is a generally decreasing function of ω it never becomes identically zero, indicating that the rectangular pulse function contains frequency components at all frequencies.

The function $(\sin x)/x = 0$ when the argument $x = n\pi$ for any integer n $(n \neq 0)$. The main peak or "lobe" of the spectrum $F(j\omega)$ is therefore contained within the frequency band defined by the first two zero-crossings $|\omega T/2| < \pi$ or $|\omega| < 2\pi/T$. Thus as the pulse duration T is decreased, the spectral bandwidth of the pulse increases as shown in Figure 15, indicating that short duration pulses have a relatively larger high frequency content.

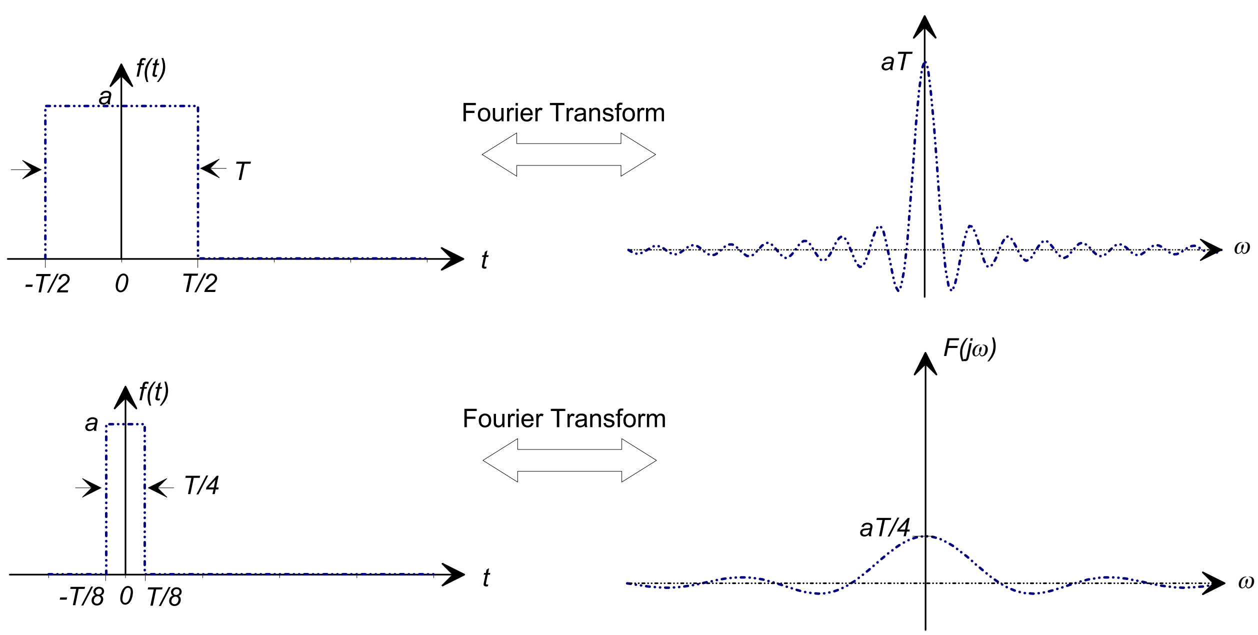

Figure 15: Dependence of the bandwidth of a pulse on its duration

■ Example

Find the Fourier transform of the Dirac delta function $\delta(t)$.

Solution: The delta impulse function, is an important theoretical function in system dynamics, defined as

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0, \end{cases} \tag{74}$$

with the additional defining property that

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

The delta function exhibits a "sifting" property when included in an integrand:

$$\int_{-\infty}^{\infty} \delta(t-T) f(t) dt = f(T). \tag{75}$$

When substituted into the forward Fourier transform

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= 1 \end{aligned} \tag{76}$$

by the sifting property. The spectrum of the delta function is therefore constant over all frequencies. It is this property that makes the impulse a very useful test input for linear systems.

■ Example

Find the Fourier transform of a finite duration sinusoidal "tone-burst"

$$f(t) = \begin{cases} \sin \omega_0 t & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$

shown in Fig. 16. shown in Figure 16.

Solution: The forward Fourier transform is

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{77}$$

$$= \int_{-T/2}^{T/2} \sin(\omega_0 t) e^{-j\omega t} dt. \tag{78}$$

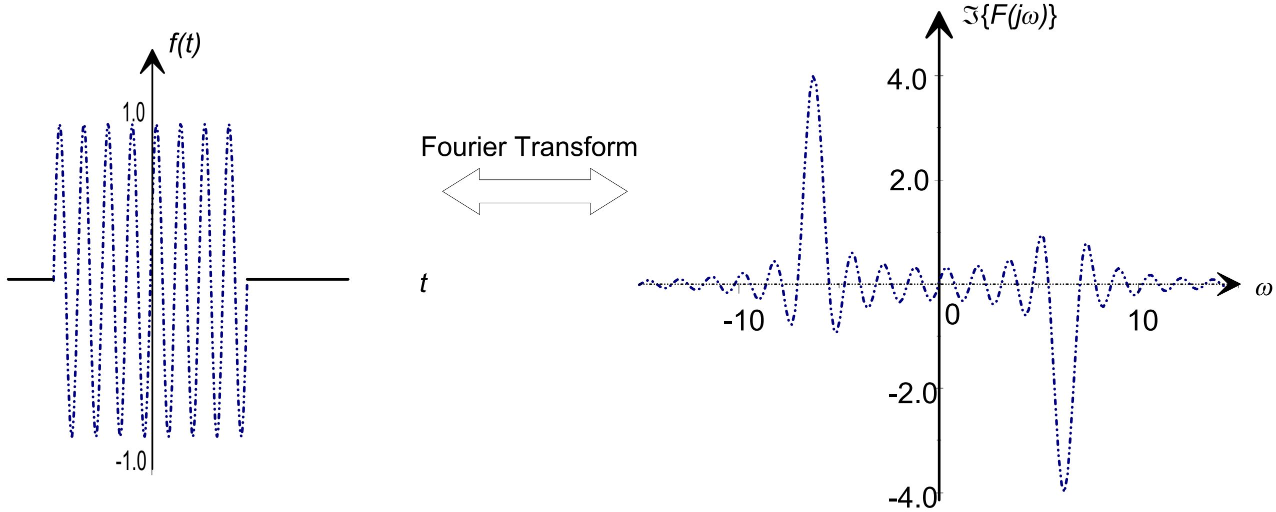

Figure 16: A sinusoidal tone-burst and its Fourier transform

The sinusoid is expanded using the Euler formula:

$$F(j\omega) = \frac{1}{j2}\int_{-T/2}^{T/2}\left[e^{-j(\omega-\omega_0)t} - e^{-j(\omega+\omega_0)t}\right]dt \quad (79)$$

$$= \frac{1}{2(\omega-\omega_0)}\left[e^{-j(\omega-\omega_0)t}\right|_{-T/2}^{T/2} - \frac{1}{2(\omega+\omega_0)}\left[e^{-j(\omega+\omega_0)t}\right|_{-T/2}^{T/2} \quad (80)$$

$$= -j\frac{T}{2}\left\{\frac{\sin\left((\omega-\omega_0)T/2\right)}{(\omega-\omega_0)T/2} - \frac{\sin\left((\omega+\omega_0)T/2\right)}{(\omega+\omega_0)T/2}\right\} \quad (81)$$

which is a purely imaginary odd function that is the sum of a pair of shifted imaginary $(\sin x)/x$ functions, centered on frequencies $\pm\omega_0$, as shown in Fig. 16. The zero-crossings of the main lobe of each function are at $\omega_0 \pm 2\pi/T$ indicating again that as the duration of a transient waveform is decreased its spectral width increases. In this case notice that as the duration T is increased the spectrum becomes narrower, and in the limit as $T \to \infty$ it achieves zero width and becomes a simple line spectrum.

■ Example

Find the Fourier transform of the one-sided real exponential function

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

(for $a > 0$) as shown in Figure 17.

Solution: From the definition of the forward Fourier transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad (82)$$

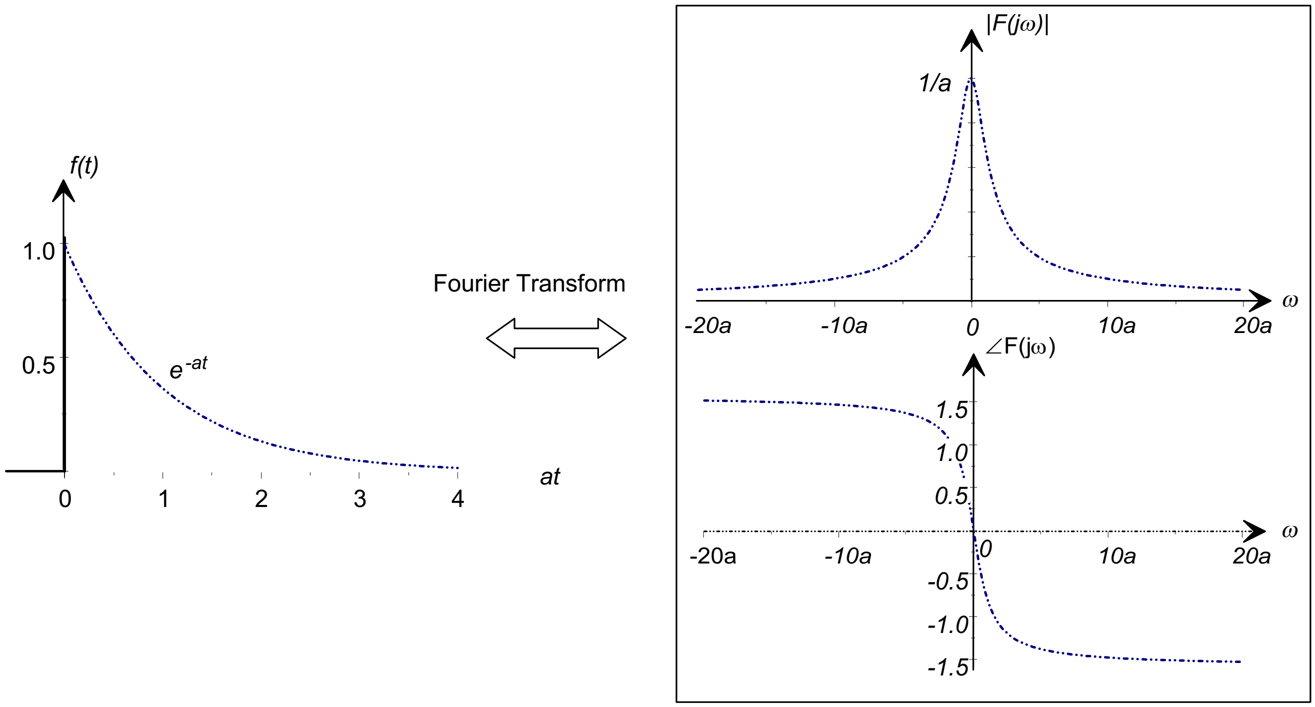

Figure 17: The one-sided real exponential function and its spectrum

$$
\begin{aligned}
&= \int_0^{\infty} e^{-at} e^{-j\omega t} dt && (83)\\
&= \left[\frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_0^{\infty} && (84)\\
&= \frac{1}{a+j\omega} && (85)
\end{aligned}
$$

which is complex, and in terms of a magnitude and phase function is

$$
\begin{aligned}
|F(j\omega)| &= \frac{1}{\sqrt{a^2+\omega^2}} && (86)\\
\angle F(j\omega) &= \tan^{-1}\left(\frac{-\omega}{a}\right) && (87)
\end{aligned}
$$

■ Example

Find the Fourier transform of a damped one-sided sinusoidal function

$$
f(t) = \begin{cases} 0 & t < 0 \\ e^{-\sigma t} \sin \omega_0 t & t \geq 0. \end{cases}
$$

(for $\sigma > 0$) as shown in Figure 18.

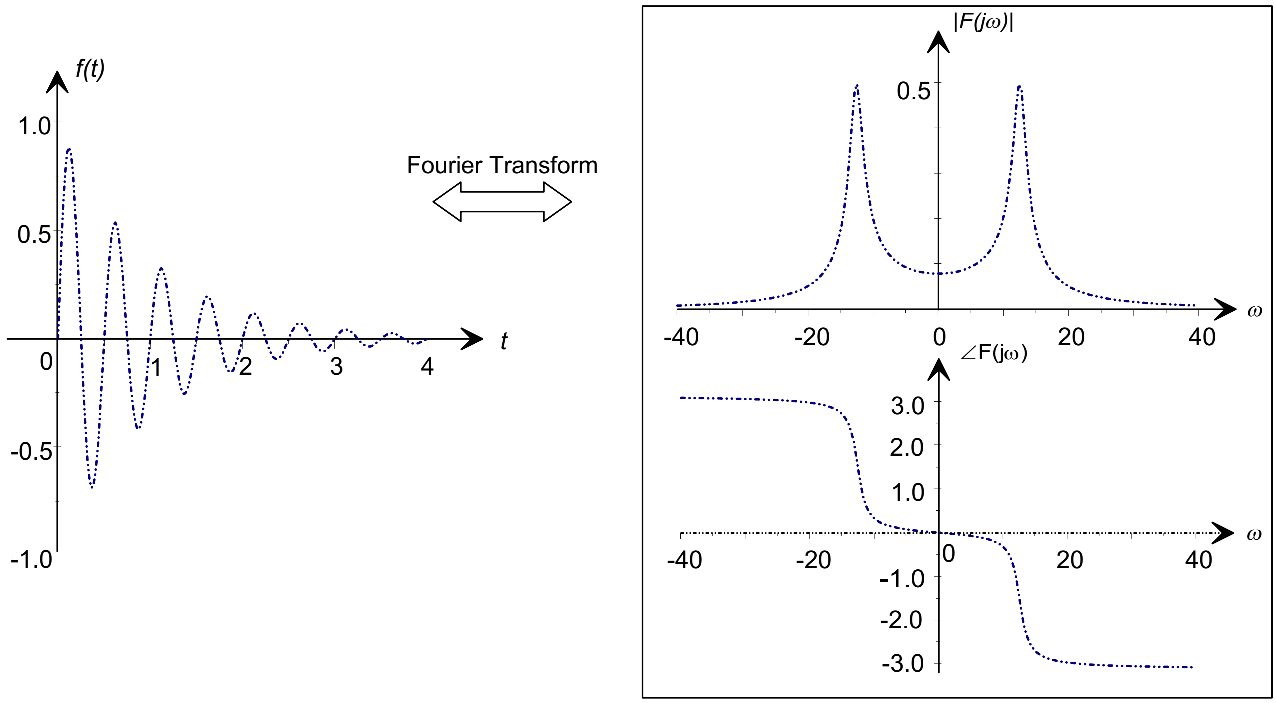

Figure 18: A damped sinusoidal function and its spectrum

Solution: From the definition of the forward Fourier transform

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt && (88) \\ &= \frac{1}{j2} \int_{0}^{\infty} e^{-\sigma t} \left(e^{j\omega_0 t} - e^{-j\omega_0 t} \right) e^{-j\omega t} dt && (89) \\ &= \frac{1}{j2} \left[- \frac{1}{\sigma + j(\omega_0 - \omega)} e^{-(\sigma + j(\omega - \omega_0))t} \right|_0^{\infty} - && (90) \\ & \qquad \frac{1}{j2} \left[- \frac{1}{\sigma + j(\omega_0 + \omega)} e^{-(\sigma + j(\omega + \omega_0))t} \right|_0^{\infty} && (91) \\ &= \frac{\omega_0}{(\sigma + j\omega)^2 + \omega_0^2}. && (92) \end{aligned}$$

The magnitude and phase of this complex quantity are

$$\begin{aligned} |F(j\omega)| &= \frac{\omega_0}{\sqrt{(\sigma^2 + \omega_0^2 - \omega^2)^2 + (2\sigma\omega)^2}} && (93) \\ \angle F(j\omega) &= \tan^{-1} \frac{-2\sigma\omega}{\sigma^2 + \omega_0^2 - \omega^2} && (94) \end{aligned}$$

4.2 Properties of the Fourier Transform

A full description of the properties of the Fourier transform is beyond the scope and intent of this book, and the interested reader is referred to the many texts devoted to the Fourier transform [2,4,5]. The properties listed below are presented because of their importance in the study of system dynamics:

(1) Existence of the Fourier Transform A modified form of the three Dirichelet conditions presented for the Fourier series guarantees the existence of the Fourier transform. These are sufficient conditions, but are not strictly necessary. For the Fourier transform the conditions are

- The function f(t) must be *integrable in the absolute sense* over all time, that is
$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty.$$
- There must be at most a finite number of maxima and minima in the function $f(t)$. Notice that periodic functions are excluded by this and the previous condition.
- There must be at most a finite number of discontinuities in the function $f(t)$, and all such discontinuities must be finite in magnitude.

(2) Linearity of the Fourier Transform Like the Fourier series, the Fourier transform is a linear operation. If two functions of time $g(t)$ and $h(t)$ have Fourier transforms $G(j\omega)$ and $H(j\omega)$, that is

$$\begin{aligned} g(t) &\overset{\mathcal{F}}{\Longleftrightarrow} G(j\omega) \\ h(t) &\overset{\mathcal{F}}{\Longleftrightarrow} H(j\omega) \end{aligned}$$

and a third function $f(t) = ag(t) + bh(t)$, where a and b are constants, then the Fourier transform of $f(t)$ is

$$F(j\omega) = aG(j\omega) + bH(j\omega). \tag{95}$$

(3) Even and Odd Functions The Fourier transform of an even function of time is a purely real function, the transform of an odd function is an imaginary function. Recall that the Fourier transform shows conjugate symmetry, that is

$$F(j\omega) = \overline{F(-j\omega)}. \tag{96}$$

or

$$\Re\left[F(j\omega)\right] = \Re\left[F(-j\omega)\right] \tag{97}$$
$$\Im\left[F(j\omega)\right] = -\Im\left[F(-j\omega)\right], \tag{98}$$

therefore the Fourier transform of an even function is both real and even, while the transform of an odd function is both imaginary and odd.

(4) Time Shifting Let $f(t)$ be a waveform with a Fourier transform $F(j\omega)$, that is

$$\mathcal{F}\{f(t)\} = F(j\omega)$$

then the Fourier transform of $f(t+\tau)$, a time shifted version of $f(t)$, is

$$\mathcal{F}\{f(t+\tau)\} = e^{j\omega\tau}F(j\omega).$$

This result can be shown easily, since by definition

$$\mathcal{F}\{f(t+\tau)\} = \int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega t}dt.$$

If the variable $\nu = t + \tau$ is substituted in the integral,

$$\begin{aligned}\mathcal{F}\{f(t+\tau)\} &= \int_{-\infty}^{\infty} f(\nu) e^{-j\omega(\nu-\tau)} d\nu \\ &= e^{j\omega\tau} \int_{-\infty}^{\infty} f(\nu) e^{-j\omega\nu} d\nu \\ &= e^{j\omega\tau} F(j\omega). \end{aligned} \tag{99}$$

If $F(j\omega)$ is expressed in polar form, having a magnitude and phase angle, this relationship may be rewritten as

$$\mathcal{F}\{f(t+\tau)\} = |F(j\omega)|\, e^{j(\angle F(j\omega)+\omega\tau)} \tag{100}$$

which indicates that the Fourier transform of a time shifted waveform has the same magnitude function as the original waveform, but has an additional phase-shift term that is directly proportional to frequency.

(5) Waveform Energy We have asserted that the time domain representation $f(t)$ and the frequency domain representation $F(j\omega)$ are both complete descriptions of the function related through the Fourier transform

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(j\omega).$$

If we consider the function $f(t)$ to be a system through or across-variable, the instantaneous power that is dissipated in a D-type element with a value of unity is equal to the square of its instantaneous value. For example, the power dissipated when voltage $v(t)$ is applied to an electrical resistance of 1 ohm is $v^2(t)$. The power associated with a complex variable $v(t)$ is $|v(t)|^2$. The "energy" of an aperiodic function in the time domain is defined as the integral of this hypothetical instantaneous power over all time

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt \tag{101}$$

Parseval's theorem [3] asserts the equivalence of the total waveform energy in the time and frequency domains by the relationship

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2 \, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\overline{F(j\omega)} d\omega. \end{aligned} \tag{102}$$

In other words, the quantity $|F(j\omega)|^2$ is a measure of the energy of the function per unit bandwidth. The energy ΔE contained between two frequencies ω_1 and ω_2 is

$$\Delta E = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} F(j\omega)\overline{F(j\omega)} d\omega. \tag{103}$$

Notice that this is a one-sided energy content and that because the Fourier transform is a two-sided spectral representation, the total energy in a real function includes contributions from both positive and negative frequencies. The function

$$\Phi(j\omega) = |F(j\omega)|^2$$

is a very important quantity in experimental system dynamics and is known as the *energy density* spectrum. It is a real function, with units of energy per unit bandwidth, and shows how the energy of a waveform $f(t)$ is distributed across the spectrum.

(6) Relationship Between the Fourier Transform and the Fourier Series of a Periodic Extension of an Aperiodic Function Let $f(t)$ be a function, with Fourier transform $F(j\omega)$, that exists only in a defined interval $|t| < \Delta/2$ centered on the time origin. Then if $f_p(t)$ is a periodic extension of $f(t)$, formed by repeating $f(t)$ at intervals $T > \Delta$, each period contains $f(t)$. Then the Fourier coefficients describing $f_p(t)$ are

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f_p(t)e^{-jn\omega_0 t}dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} f(t)e^{-jn\omega_0 t}dt \\ &= \frac{1}{T}F(jn\omega_0). \end{aligned} \tag{104}$$

The Fourier coefficients of a periodic function are scaled samples of the Fourier transform of the function contained within a single period. The transform thus forms the envelope function for the definition of the Fourier series as discussed in Section 2.

(7) The Fourier Transform of the Derivative of a Function If a function $f(t)$ has a Fourier transform $F(j\omega)$ then

$$\mathcal{F}\left\{\frac{df}{dt}\right\} = j\omega F(j\omega),$$

which is easily shown using integration by parts:

$$\begin{aligned} \mathcal{F}\left\{\frac{df}{dt}\right\} &= \int_{-\infty}^{\infty} \frac{df}{dt}e^{-j\omega t}dt \\ &= \left|f(t)e^{-j\omega t}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-j\omega)e^{-j\omega t}dt \\ &= 0 + j\omega F(j\omega) \end{aligned}$$

since by definition $f(t) = 0$ at $t = \pm\infty$.

This result can be applied repetitively to show that the Fourier transform of the nth derivative of $f(t)$ is

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}\right\} = (j\omega)^n F(j\omega) \tag{105}$$

5 Fourier Transform Based Properties of Linear Systems

5.1 Response of Linear Systems to Aperiodic Inputs

The Fourier transform provides an alternative method for computing the response of a linear system to a transient input. Assume that a linear system with frequency response $H(j\omega)$ is initially at rest, and is subsequently subjected to an aperiodic input $u(t)$ having a Fourier Transform $U(j\omega)$. The task is to compute the response $y(t)$.

Assume that the input function $u(t)$, is a periodic function $u_p(t)$ with an arbitrarily large period T, and that since $u_p(t)$ is periodic it can be described by a set of complex Fourier coefficients $\{U_n\}$. The output $y_p(t)$ is periodic with period T and is described by its own set of Fourier coefficients $\{Y_n\}$. In Section 3.2 it was shown that the output coefficients are

$$Y_n = H(jn\omega_0)U_n, \tag{106}$$

and that the Fourier synthesis equation for the output is

$$\begin{aligned} y_p(t) &= \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t} & (107) \\ &= \sum_{n=-\infty}^{\infty} H(jn\omega_0) U_n e^{jn\omega_0 t} & (108) \\ &= \sum_{n=-\infty}^{\infty} H(jn\omega_0) \left\{ \frac{\omega_0}{2\pi} \int_{-T/2}^{T/2} u_p(t) e^{-jn\omega_0 t} dt \right\} e^{jn\omega_0 t} & (109) \end{aligned}$$

As in the development of the Fourier transform, we let $T \to \infty$ so that $u_p(t) \to u(t)$, and in the limit write the summation as an integral

$$\begin{aligned} y(t) &= \lim_{T\to\infty} y_p(t) \\ &= \int_{-\infty}^{\infty} H(j\omega) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt \right\} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) U(j\omega) e^{j\omega t} d\omega. & (110) \end{aligned}$$

This equation expresses the system output $y(t)$ in the form of the inverse Fourier transform of the product $H(j\omega)U(j\omega)$ or

$$y(t) = \mathcal{F}^{-1}\left\{H(j\omega)U(j\omega)\right\}. \tag{111}$$

We can therefore write

$$Y(j\omega) = H(j\omega)U(j\omega), \tag{112}$$

which is the fundamental frequency-domain input/output relationship for a linear system. The output spectrum is therefore the product of the input spectrum and the system frequency response function:

> Given a relaxed linear system with a frequency response $H(j\omega)$ and an input that possesses a Fourier transform, the response may be found by the following three steps:
>
> **(1)** Compute the Fourier transform of the input
>
> $$U(j\omega) = \mathcal{F}\left\{u(t)\right\}.$$
>
> **(2)** Form the output spectrum as the product
>
> $$Y(j\omega) = H(j\omega)U(j\omega),$$
>
> **(3)** Compute the inverse Fourier transform
>
> $$y(t) = \mathcal{F}^{-1}\left\{Y(j\omega)\right\}.$$

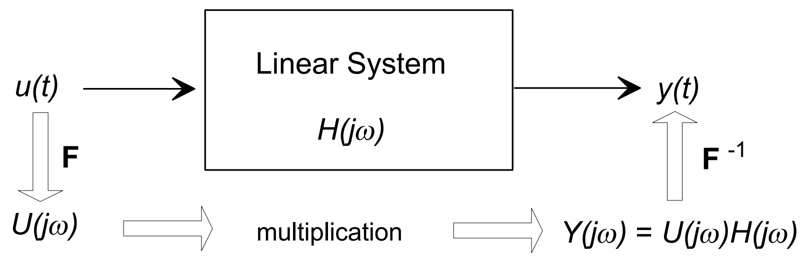

Figure 19: Frequency domain computation of system response.

Figure 19 illustrates the steps involved in computing the system response using the Fourier transform method.

■ Example

Use the Fourier transform method to find the response of a linear first-order system with a transfer function

$$H(s) = \frac{1}{\tau s + 1}$$

to a one-sided decaying exponential input

$$u(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

where $a > 0$.

Solution: The frequency response of the system is

$$H(j\omega) = \frac{1/\tau}{\tau + j\omega} \tag{113}$$

and from Example 4.1 the Fourier transform of a decaying exponential input is

$$\begin{aligned} U(j\omega) &= \mathcal{F}\{u(t)\} \\ &= \mathcal{F}\{e^{-a\tau}\} \\ &= \frac{1}{a + j\omega}. \end{aligned} \tag{114}$$

The output spectrum is the product of the transfer function and the frequency response

$$\begin{aligned} Y(j\omega) &= H(j\omega)U(j\omega) \\ &= \frac{1/\tau}{1/\tau + j\omega} . \frac{1}{a + j\omega}. \end{aligned} \tag{115}$$

In order to compute the time domain response through the inverse transform, it is convenient to expand $Y(j\omega)$ in terms of its partial fractions

$$Y(j\omega) = \frac{1}{a\tau - 1}\left[\frac{1}{1/\tau + j\omega} - \frac{1}{a + j\omega}\right] \tag{116}$$

provided $a \neq 1/\tau$, and using the linearity property of the inverse transform

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{Y(j\omega)\} \\ &= \frac{1}{a\tau - 1}\left[\mathcal{F}^{-1}\left\{\frac{1}{1/\tau + j\omega}\right\} - \mathcal{F}^{-1}\left\{\frac{1}{a + j\omega}\right\}\right] \end{aligned} \tag{117}$$

Using the results of Example 4.1 once more

$$e^{-a\tau} \stackrel{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a + j\omega}, \tag{118}$$

the desired solution is

$$y(t) = \frac{1}{a\tau - 1}\left[e^{-t/\tau} - e^{-at}\right] \tag{119}$$

These input/output relationships are summarized in Figure 20.

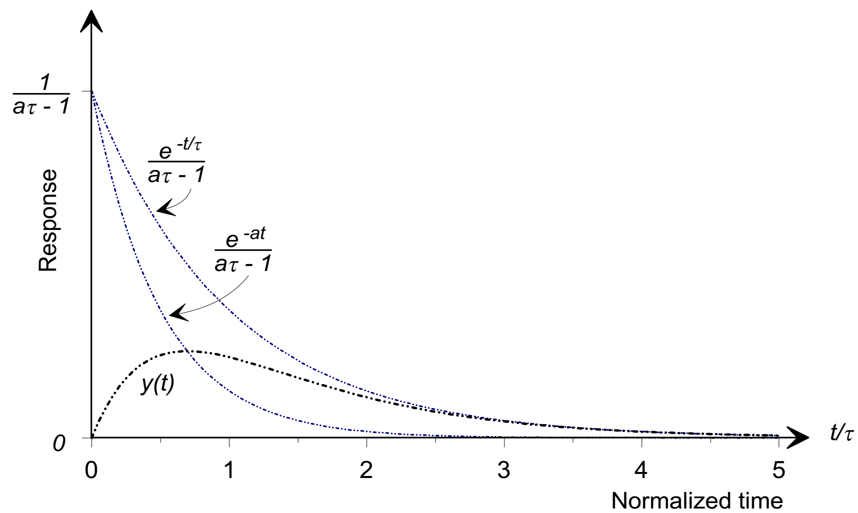

Figure 20: First-order system response to an exponential input

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Before the 1960's frequency domain analysis methods were of theoretical interest, but the difficulty of numerically computing Fourier transforms limited their applicability to experimental studies. The problem lay in the fact that numerical computation of the transform of n samples of data required n^2 complex multiplications, which took an inordinate amount of time on existing digital computers. In the 1960's a set of computational algorithms, known as the Fast Fourier transform (FFT) methods, that required only $n \log_2 n$ multiplications for computing the Fourier transform of experimental data were developed. The computational savings are very great, for example in order to compute the transform of 1024 data points, the FFT algorithm is faster by a factor of more than 500. These computational procedures revolutionized spectral analysis and frequency domain analysis of system behavior, and opened up many new analysis methods that had previously been impractical. FFT based system analysis is now routinely done in both software and in dedicated digital signal-processing (DSP) electronic hardware. These techniques are based on a "discrete-time" version of the continuous Fourier transforms described above, and have some minor differences in definition and interpretation.

5.2 The Frequency Response Defined Directly from the Fourier Transform

The system frequency response function $H(j\omega)$ may be defined directly using the transform property of derivatives. Consider a linear system described by the single input/output differential equation

$$a_n\frac{d^ny}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\ldots+a_1\frac{dy}{dt}+a_0y = b_m\frac{d^mu}{dt^m}+b_{m-1}\frac{d^{m-1}u}{dt^{m-1}}+\ldots+b_1\frac{du}{dt}+b_0u \tag{120}$$

and assume that the Fourier transforms of both the input $u(t)$ and the output $y(t)$ exist. Then the Fourier transform of both sides of the differential equation may be found by using the derivative property (Property (7) of Section 4.2):

$$\mathcal{F}\left\{\frac{d^nf}{dt^n}\right\}=(j\omega)^nF(j\omega)$$

to give

$$\left\{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_1(j\omega)+a_0\right\}Y(j\omega) = \left\{b_m(j\omega)^m+b_{m-1}(j\omega)^{m-1}+\ldots+b_1(j\omega)+b_0\right\}U(j\omega), \tag{121}$$

which has reduced the original differential equation to an algebraic equation in $j\omega$. This equation may be rewritten explicitly in terms of $Y(j\omega)$ in terms of the frequency response $H(j\omega)$

$$Y(j\omega) = \frac{b_m(j\omega)^m+b_{m-1}(j\omega)^{m-1}+\ldots+b_1(j\omega)+b_0}{a_n(j\omega)^n+a_{n-1}(j\omega)^{n-1}+\ldots+a_1(j\omega)+a_0}U(j\omega) \tag{122}$$

$$= H(j\omega)U(j\omega), \tag{123}$$

showing again the generalized multiplicative frequency domain relationship between input and output.

5.3 Relationship between the Frequency Response and the Impulse Response

In Example 4.1 it is shown that the Dirac delta function $\delta(t)$ has a unique property; its Fourier transform is unity for all frequencies

$$\mathcal{F}\{\delta(t)\}=1,$$

The impulse response of a system $h(t)$ is defined to be the response to an input $u(t)=\delta(t)$, the output spectrum is then $Y_\delta(j\omega)=\mathcal{F}\{h(t)\}$,

$$Y(j\omega) = \mathcal{F}\{\delta(t)\}H(j\omega)$$

$$= H(j\omega). \tag{124}$$

or

$$h(t)=\mathcal{F}^{-1}\{H(j\omega)\}. \tag{125}$$

In other words, the system impulse response $h(t)$ and its frequency response $H(j\omega)$ are a Fourier transform pair:

$$h(t)\overset{\mathcal{F}}{\Longleftrightarrow}H(j\omega). \tag{126}$$

In the same sense that $H(j\omega)$ completely characterizes a linear system in the frequency response, the impulse response provides a complete system characterization in the time domain.

■ Example

An unknown electrical circuit is driven by a pulse generator, and its output is connected to a recorder for subsequent analysis, as shown in Figure 21.

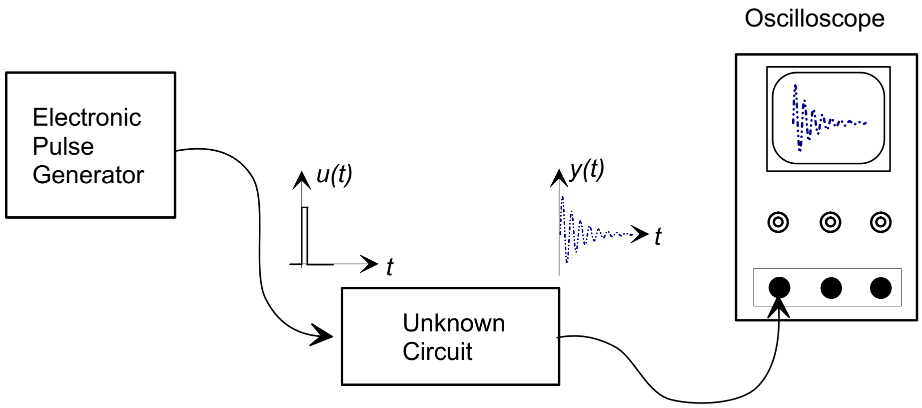

Figure 21: Setup for estimating the frequency response of an electrical circuit

The pulse generator produces pulses of 1 msec. duration and an amplitude of 10 volts. When the circuit is excited by a single pulse the output is found to be very closely approximated by a damped sinusoidal oscillation of the form

$$y(t) = 0.02e^{-t}sin(10t).$$

Estimate the frequency response of the system.

Solution: The input $u(t)$ is a short rectangular pulse, much shorter in duration than the observed duration of the system response. The impulse function $\delta(t)$ is the limiting case of a unit area rectangular pulse, as its duration approached zero. For this example we assume that the duration of the pulse is short enough to approximate a delta function, and because this pulse has an area of $10 \times 0.001 = 0.01$ v-s, we assume

$$u(t) = 0.01\delta(t) \tag{127}$$

and therefore assume that the observed response is a scaled version of the system impulse response,

$$y(t) = 0.01h(t), \tag{128}$$

or

$$\begin{aligned} h(t) &= 100y(t) \\ &= 2e^{-5t}sin(12t). \end{aligned}$$

The frequency response is

$$\begin{aligned} H(j\omega) &= \mathcal{F}\{h(t)\} \\ &= 2\mathcal{F}\left\{e^{-5t}sin(12t)\right\}. \end{aligned} \tag{129}$$

In Example 4.1 it is shown that

$$\mathcal{F}\left\{e^{-\sigma t}\sin\omega_0 t\right\} = \frac{\omega_0}{(\sigma+j\omega)^2+\omega_0^2},$$

and substituting $\omega_0 = 12$, $\sigma = 5$ gives

$$H(j\omega) = \frac{24}{(j\omega)^2 + j20\omega + 169} \tag{130}$$

We therefore make the substitution $s = j\omega$ and conclude that our unknown electrical network is a second-order system with a transfer function

$$H(s) = \frac{24}{s^2 + 20s + 169}, \tag{131}$$

which has an undamped natural frequency $\omega_n = 13$ radians/sec. and a damping ratio $\zeta = 10/13$. The input/output differential equation is

$$\frac{d^2y}{dt^2} + 20\frac{dy}{dt} + 169y = 24u(t). \tag{132}$$

5.4 The Convolution Property

A system with an impulse response $h(t)$, driven by an input $u(t)$, responds with an output $y(t)$ given by the convolution integral

$$\begin{aligned} y(t) &= h(t)\star u(t) \\ &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \end{aligned} \tag{133}$$

or alternatively by changing the variable of integration

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau. \tag{134}$$

In the frequency domain the input/output relationship for a linear system is multiplicative, that is $Y(j\omega) = U(j\omega)H(j\omega)$. Because by definition

$$y(t) \stackrel{\mathcal{F}}{\Longleftrightarrow} Y(j\omega),$$

we are lead to the conclusion that

$$h(t)\star u(t) \stackrel{\mathcal{F}}{\Longleftrightarrow} H(j\omega)U(j\omega). \tag{135}$$

The computationally intensive operation of computing the convolution integral has been replaced by the operation of multiplication. This result, known as the convolution property of the Fourier transform, can be shown to be true for the product of any two spectra, for example $F(j\omega)$ and $G(j\omega)$

$$\begin{aligned} F(j\omega)G(j\omega) &= \int_{-\infty}^{\infty} f(\nu)e^{-j\omega\nu}d\nu.\int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\nu)g(\tau)e^{-j\omega(\nu+\tau)}d\tau d\nu, \end{aligned}$$

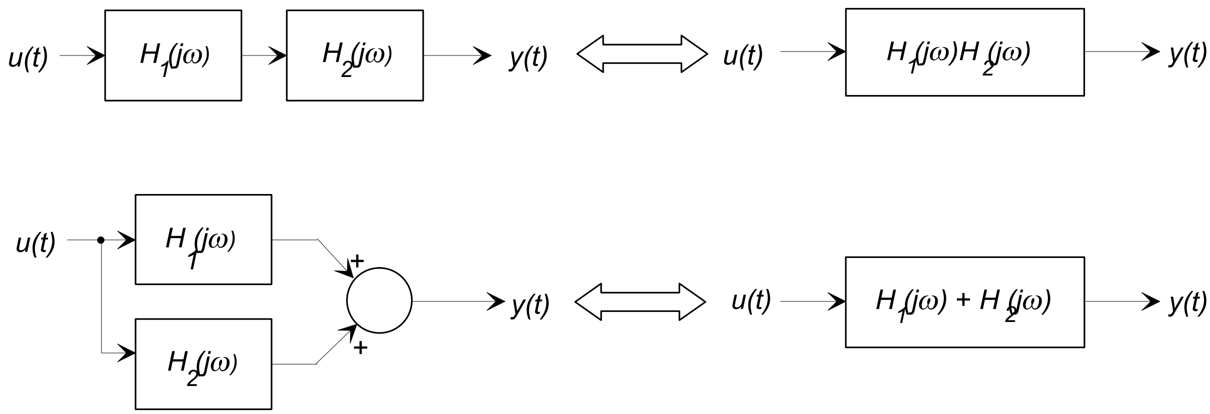

Figure 22: Frequency Response of Cascaded and Parallel Linear Systems

and with the substitution $t = \nu + \tau$

$$\begin{aligned} H(j\omega)U(j\omega) &= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau\right\} e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \left(f(t) \star g(t)\right) e^{-j\omega t}dt \\ &= \mathcal{F}\left\{f(t) \star g(t)\right\}. \end{aligned} \tag{136}$$

A *dual* property holds: if any two functions, $f(t)$ and $g(t)$, are multiplied together in the time domain, then the Fourier transform of their product is a convolution of their spectra. The dual convolution/multiplication properties are

$$f(t) \star g(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(j\omega)G(j\omega) \tag{137}$$

$$f(t)g(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi}F(j\omega) \star G(j\omega). \tag{138}$$

5.5 The Frequency Response of Interconnected Systems

If two linear systems $H_1(j\omega)$ and $H_2(j\omega)$ are connected in cascade, or series, as shown if Fig. 22 so that the output variable of the first is the input to the second, then provided the interconnection does not affect $y_1(t)$, overall frequency response is

$$\begin{aligned} Y_2(j\omega) &= H_2(j\omega)Y_1(j\omega) \\ &= H_2(j\omega)\left\{H_1(j\omega)U(j\omega)\right\} \\ &= \left\{H_2(j\omega)H_1(j\omega)\right\}U(j\omega) \end{aligned} \tag{139}$$

The overall frequency response is therefore the product of the two cascaded frequency responses

$$H(j\omega) = H_1(j\omega)H_2(j\omega). \tag{140}$$

Similarly, if two linear systems are connected in parallel so that their outputs are summed together, then

$$\begin{aligned} Y(j\omega) &= H_1(j\omega)U(j\omega) + H_2(j\omega)U(j\omega) \\ &= \left\{H_1(j\omega) + H_2(j\omega)\right\}U(j\omega), \end{aligned} \tag{141}$$

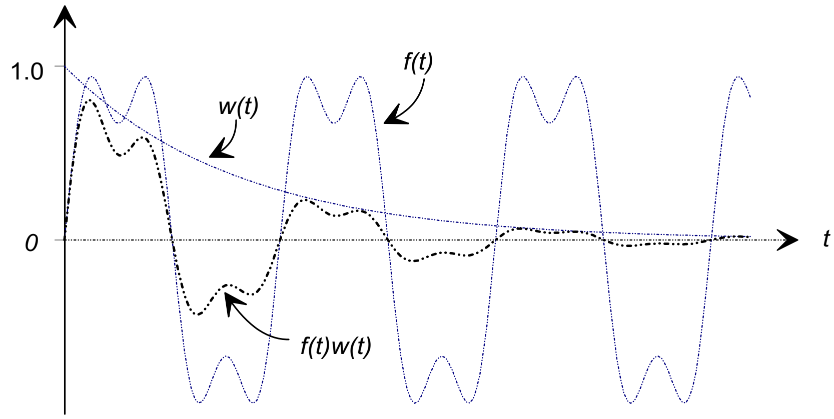

Figure 23: Modification of a function by a multiplicative weighting function

so that the overall frequency response is the sum of the two component frequency responses

$$H(j\omega) = H_1(j\omega) + H_2(j\omega). \tag{142}$$

6 The Laplace Transform

While the Fourier transform is an important theoretical and practical tool for the analysis and design of linear systems, there are classes of waveforms for which the integral defining the transform does not converge. Two important functions that do not have Fourier transforms are the unit step function

$$u_s(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0, \end{cases}$$

and the ramp function

$$r(t) = \begin{cases} 0 & t \leq 0, \\ t & t > 0. \end{cases}$$

Neither of these functions is integrable in the absolute sense, for example

$$\int_{-\infty}^{\infty} |u_s(t)|\, dt = \infty,$$

and the forward Fourier transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

does not converge for either function. The Laplace transform is a generalized form of the Fourier transform that exists for a much broader range of functions.

The development of the Fourier transform, described in Section 3, requires that the time function $f(t)$ is limited in duration and can be described by a Fourier series of a periodic extension of the waveform. Neither the step nor the ramp function satisfies this condition; they are representative of a broad range of functions that are unlimited in extent. The Laplace transform of $f(t)$ is the Fourier transform of a modified function, formed by multiplying $f(t)$ by a *weighting* function $w(t)$ that forces the product $f(t)w(t)$ to zero as time t becomes large. In particular, the Laplace transform uses an exponential weighting function

$$w(t) = e^{-\sigma t} \tag{143}$$

where σ is real. Figure 23 shows how this function will force the product $w(t)f(t)$ to zero for large values of t. Then for a given value of σ, provided

$$\int_{-\infty}^{\infty} |w(t)f(t)|\, dt < \infty,$$

the Fourier transform of $f(t)e^{-\sigma t}$:

$$F(j\omega|\sigma) = \mathcal{F}\left\{f(t)e^{-\sigma t}\right\} = \int_{-\infty}^{\infty} (f(t)e^{-\sigma t})e^{-j\omega t} dt \tag{144}$$

will exist. The modified transform is not a function of angular frequency ω alone, but also of the value of the weighting constant σ. The Laplace transform combines both ω and σ into a single complex variable s

$$s = \sigma + j\omega \tag{145}$$

and defines the *two-sided* transform as a function of the complex variable s

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} (f(t)e^{-\sigma t})e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-st} dt \end{aligned} \tag{146}$$

For a given $f(t)$ the integral may converge for some values of σ but not others. The *region of convergence* (ROC) of the integral in the complex s-plane is an important qualification that should be specified for each transform $F(s)$. Notice that when $\sigma = 0$, so that $w(t) = 1$, the Laplace transform reverts to the Fourier transform. Thus, if $f(t)$ has a Fourier transform

$$F(j\omega) = F(s)\,|_{s=j\omega}\,. \tag{147}$$

Stated another way, a function $f(t)$ has a Fourier transform if the region of convergence of the Laplace transform in the s-plane includes the imaginary axis.

In engineering analyses it is usual to restrict the application of the Laplace transform to those functions for which $f(t) = 0$ for $t < 0$. Under this restriction the integrand is zero for all negative time and the limits on the integral may be changed

$$F(s) = \int_{0}^{\infty} f(t)e^{-st} dt, \tag{148}$$

which is commonly known as the *one-sided* Laplace transform. In this book we discuss only the properties and use of this one-sided transform, and refer to it as the Laplace transform. It should be kept clearly in mind that the requirement

$$f(t) = 0 \qquad \text{for } t < 0$$

must be met in order to satisfy the definition of the Laplace transform.

The inverse Laplace transform may be defined from the Fourier transform. Since

$$F(s) = F(\sigma + j\omega) = \mathcal{F}\left\{f(t)e^{-\sigma t}\right\}$$

the inverse Fourier transform of $F(s)$ is

$$f(t)e^{-\sigma t} = \mathcal{F}\left\{F(s)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}d\omega. \tag{149}$$

If each side of the equation is multiplied by $e^{\sigma t}$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{st}d\omega. \tag{150}$$

The variable of integration may be changed from ω to $s = \sigma + j\omega$, so that $ds = jd\omega$, and with the corresponding change in the limits the inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds \tag{151}$$

The evaluation of this integral requires integration along a path parallel to the $j\omega$ axis in the complex s plane. As will be shown below, it is rarely necessary to compute the inverse Laplace transform in practice.

The one-sided Laplace transform pair is defined as

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{152}$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds. \tag{153}$$

The equations are a transform pair in the sense that it is possible to move uniquely between the two representations. The Laplace transform retains many of the properties of the Fourier transform and is widely used throughout engineering systems analysis.

We adopt a nomenclature similar to that used for the Fourier transform to indicate Laplace transform relationships between variables. Time domain functions are designated by a lower-case letter, such as y(t), and the frequency domain function use the same upper-case letter, $Y(s)$. For one-sided waveforms we differentiation between the Laplace and Fourier transforms by the argument $F(s)$ or $F(j\omega)$ on the basis that

$$F(j\omega) = F(s)|_{s=j\omega}$$

A bidirectional Laplace transform relationship between a pair of variables is indicated by the nomenclature

$$f(t) \stackrel{\mathcal{L}}{\Longleftrightarrow} F(s),$$

and the operations of the forward and inverse Laplace transforms are written:

$$\begin{aligned} \mathcal{L}\left\{f(t)\right\} &= F(s) \\ \mathcal{L}^{-1}\left\{F(s)\right\} &= f(t). \end{aligned}$$

6.1 Laplace Transform Examples

■ Example

Find the Laplace transform of the unit step function

$$u_s(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0. \end{cases}$$

Solution: From the definition of the Laplace transform

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}dt &(154)\\ &= \int_0^\infty e^{-st}dt \\ &= \left[-\frac{1}{s}e^{-st}\right|_0^\infty \\ &= \frac{1}{s}, &(155) \end{aligned}$$

provided $\sigma > 0$. Notice that the integral does not converge for $\sigma = 0$, and therefore that the unit step does not have a Fourier transform.

■ Example

Find the Laplace transform of the one-sided real exponential function

$$f(t) = \begin{cases} 0 & t \leq 0 \\ e^{at} & t > 0. \end{cases}$$

Solution: In Example 4.1 the Fourier transform of a real exponential waveform with a negative exponent was found. In this example we let the exponent be positive or negative.

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}dt &(156)\\ &= \int_0^\infty e^{-(s-a)t}dt \\ &= \left[-\frac{1}{s-a}e^{-(s-a)t}\right|_0^\infty \\ &= \frac{1}{s-a} &(157) \end{aligned}$$

The integral will converge only if $\sigma > a$ and therefore the region of convergence is all of the s-plane to the right of $\sigma = a$.

■ Example

Find the Laplace transform of the one-sided ramp function

$$f(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0. \end{cases}$$

Solution: The ramp function does not posses a Fourier transform, but its Laplace transform is

$$F(s) = \int_0^\infty te^{-st}dt, \tag{158}$$

and integrating by parts

$$\begin{aligned} F(s) &= \left[-\frac{1}{s}te^{-st} \right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \qquad (159) \\ &= 0 + \frac{1}{s^2}\left[e^{-st} \right|_0^\infty \\ &= \frac{1}{s^2} \qquad (160) \end{aligned}$$

The region of convergence is all of the s-plane to the right of $\sigma = 0$, that is the right half-plane.

■ Example

Find the Laplace transform of the Dirac delta function $\delta(t)$. In Example 4.1 it was shown that $\delta(t)$ had the important property that $\mathcal{F}\{\delta(t)\} = 1$.

Solution: When substituted into the Laplace transform

$$\begin{aligned} \Delta(s) &= \int_0^\infty \delta(t)e^{-st}dt \qquad (161) \\ &= 1 \qquad (162) \end{aligned}$$

by the sifting property of the impulse function. Thus $\delta(t)$ has a similar property in the Fourier and Laplace domains; its transform is unity and it converges everywhere.

■ Example

Find the Laplace transform of a one-sided sinusoidal function

$$f(t) = \begin{cases} 0 & t \le 0. \\ \sin \omega_0 t & t > 0. \end{cases}$$

Solution: The Laplace transform is

$$F(s) = \int_0^\infty \sin(\omega_0 t)e^{-st}dt, \tag{163}$$

and the sine may be expanded as a pair of complex exponentials using the Euler formula

$$\begin{aligned} F(s) &= \frac{1}{j2}\int_0^\infty \left[e^{-(\sigma+j(\omega-\omega_0))t} - e^{-(\sigma+j(\omega+\omega_0))t}\right] dt \quad (164)\\ &= \left[-\frac{1}{\sigma+j(\omega-\omega_0)}e^{-(\sigma+j(\omega-\omega_0))t}\right|_0^\infty - \left[-\frac{1}{\sigma+j(\omega+\omega_0)}e^{-(\sigma+j(\omega+\omega_0))t}\right|_0^\infty \\ &= \frac{\omega_0}{(\sigma+j\omega)^2+\omega_0^2} \\ &= \frac{\omega_0}{s^2+\omega_0^2} \quad (165)\end{aligned}$$

for all $\sigma > 0$.

These and other common Laplace transform pairs are summarized in Table 2.

6.2 Properties of the Laplace Transform

(1) Existence of the Laplace Transform The Laplace transform exists for a much broader range of functions than the Fourier transform. Provided the function $f(t)$ has a finite number of discontinuities in the interval $0 < t < \infty$, and all such discontinuities are finite in magnitude, the transform converges for $\sigma > \alpha$ provided there can be found a pair of numbers M, and α, such that

$$|f(t)| \leq Me^{\alpha t}$$

for all $t \geq 0$. As with the Dirichelet conditions for the Fourier transform, this is a sufficient condition to guarantee the existence of the integral but it is not strictly necessary.

While there are functions that do not satisfy this condition, for example $e^{t^2} > Me^{\alpha t}$ for any M and α at sufficiently large values of t, the Laplace transform does exist for most functions of interest in the field of system dynamics.

(2) Linearity of the Laplace Transform Like the Fourier transform, the Laplace transform is a linear operation. If two functions of time $g(t)$ and $h(t)$ have Laplace transforms $G(s)$ and $H(s)$, that is

$$\begin{aligned} g(t) &\stackrel{\mathcal{L}}{\Longleftrightarrow} G(s) \\ h(t) &\stackrel{\mathcal{L}}{\Longleftrightarrow} H(s) \end{aligned}$$

then

$$\mathcal{L}\left\{ag(t)+bh(t)\right\} = a\mathcal{L}\left\{g(t)\right\} + b\mathcal{L}\left\{h(t)\right\}. \quad (166)$$

which is easily shown by substitution into the transform integral.

(3) Time Shifting If $F(s) = \mathcal{L}f(t)$ then

$$\mathcal{L}\{f(t+\tau)\} = e^{s\tau}F(s). \quad (167)$$

This property follows directly from the definition of the transform

$$\mathcal{L}\left\{f(t+\tau)\right\} = \int_0^\infty f(t+\tau)e^{-st}dt$$

$f(t)$ for $t \geq 0$	$F(s)$
$\delta(t)$	1
$u_s(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^k	$\frac{k!}{s^{k+1}}$
e^{-at}	$\frac{1}{s+a}$
$t^k e^{-at}$	$\frac{k!}{(s+a)^{k+1}}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$1 + \frac{b}{a-b} e^{-at} - \frac{a}{a-b} e^{-bt}$	$\frac{ab}{a(s+a)(s+b)}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$e^{-at}\left(\omega \cos \omega t - a \sin \omega t\right)$	$\frac{\omega s}{(s+a)^2+\omega^2}$

Table 2: Table of Laplace transforms $F(s)$ of some common one-sided functions of time $f(t)$.

and if the variable of integration is changed to $\nu = t + \tau$,

$$\begin{aligned} \mathcal{L}\{f(t+\tau)\} &= \int_0^\infty f(\nu) e^{-s(\nu-\tau)} d\nu \\ &= e^{s\tau} \int_0^\infty f(\nu) e^{-s\nu} d\nu \\ &= e^{s\tau} F(s). \end{aligned} \tag{168}$$

(4) <u>**The Laplace Transform of the Derivative of a Function**</u> If a function $f(t)$ has a Laplace transform $F(s)$, the Laplace transform of the derivative of $f(t)$ is

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0). \tag{169}$$

Using integration by parts

$$\begin{aligned} \mathcal{L}\left\{\frac{df}{dt}\right\} &= \int_0^\infty \frac{df}{dt} e^{-st} dt \\ &= \left| f(t) e^{-st} \right|_0^\infty + \int_0^\infty s f(t) e^{-st} dt \\ &= sF(s) - f(0). \end{aligned}$$

This procedure may be repeated to find the Laplace transform of higher order derivatives, for example the Laplace transform of the second derivative is

$$\begin{aligned} \mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} &= s\left[s\mathcal{L}\left\{f(t)\right\} - f(0)\right] - \left.\frac{df}{dt}\right|_{t=0} \\ &= s^2 F(s) - sf(0) - \left.\frac{df}{dt}\right|_{t=0} \end{aligned} \tag{170}$$

which may be generalized to

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} \left(\left.\frac{d^{i-1} f}{dt^{i-1}}\right|_{t=0} \right) \tag{171}$$

for the n derivative of $f(t)$.

(5) <u>**The Laplace Transform of the Integral of a Function**</u> If $f(t)$ is a one-sided function of time with a Laplace transform $F(s)$, the Laplace transform of the integral of $f(t)$ is

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s). \tag{172}$$

If a new function $g(t)$, which is the integral of $f(t)$, is defined

$$g(t) = \int_0^t f(\tau) d\tau$$

then the derivative property shows that

$$\mathcal{L}\left\{f(t)\right\} = sG(s) - g(0),$$

and since $g(0) = 0$, we obtain the desired result.

$$G(s) = \frac{1}{s} F(s)$$

(6) The Laplace Transform of a Periodic Function The Laplace transform of a one-sided periodic continuous function with period $T(>0)$ is

$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt. \qquad (173)$$

Define a new function $f_1(t)$ that is defined over one period of the waveform

$$f_1(t) = \begin{cases} f(t) & 0 < t \leq T, \\ 0 & \text{otherwise} \end{cases}$$

so that $f(t)$ may be written

$$f(t) = f_1(t) + f_1(t+T) + f_1(t+2T) + f_1(t+3T) + \ldots$$

then using the time-shifting property above

$$\begin{aligned} F(s) &= F_1(s) + e^{-sT}F_1(s) + e^{-s2T}F_1(s) + e^{-s3T}F_1(s) + \ldots \\ &= \left(1 + e^{-sT} + e^{-s2T} + e^{-s3T} + \ldots\right) F_1(s) \end{aligned}$$

The quantity in parentheses is a geometric series whose sum is $1/(1-e^{-sT})$ with the desired result

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-sT}} F_1(s) \\ &= \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt \end{aligned}$$

(7) The Final Value Theorem The final value theorem relates the steady-state behavior of a time domain function $f(t)$ to its Laplace transform. It applies only if $f(t)$ does in fact settle down to a steady (constant) value as $t \to \infty$. For example a sinusoidal function $\sin \omega t$ does not have a steady-state value, and the final value theorem does not apply.

If $f(t)$ and its first derivative both have Laplace transforms, and if $\lim_{t\to\infty} f(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad (174)$$

To prove the theorem, consider the limit as s approaches zero in the Laplace transform of the derivative

$$\lim_{s\to 0} \int_0^\infty \left[\frac{d}{dt} f(t)\right] e^{-st} dt = \lim_{s\to 0} \left[sF(s) - f(0)\right]$$

from the derivative property above. Since $\lim_{s\to 0} e^{-st} = 1$

$$\begin{aligned} \int_0^\infty \left[\frac{d}{dt} f(t)\right] dt &= f(t)\,|_0^\infty \\ &= f(\infty) - f(0) \\ &= \lim_{s\to 0} sF(s) - f(0), \end{aligned}$$

from which we conclude

$$f(\infty) = \lim_{s\to 0} sF(s).$$

6.3 Computation of the Inverse Laplace Transform

Evaluation of the inverse Laplace transform integral, defined in Eq. (153), involves contour integration in the region of convergence in the complex plane, along a path parallel to the imaginary axis. In practice this integral is rarely solved, and the inverse transform is found by recourse to tables of transform pairs, such as Table 2. In systems analysis Laplace transforms usually appear as rational functions of the complex variable s, that is

$$F(s) = \frac{N(s)}{D(s)}$$

where the degree of the numerator polynomial $N(s)$ is at most equal to the degree of the denominator polynomial $D(s)$. The method of partial fractions, described in Appendix C, may be used to express F(s) as a sum of much simpler rational functions, all of which have well known inverse transforms. For example, suppose that $F(s)$ may be written in factored form

$$F(s) = \frac{K(s+b_1)(s+b_2)\dots(s+b_m)}{(s+a_1)(s+a_2)\dots(s+a_n)}$$

where $n \geq m$, and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$ are all either real or appear in complex conjugate pairs, if all of the a_i are distinct, then the transform may be written as a sum of first-order terms

$$F(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \dots + \frac{A_n}{s+a_n}$$

where the partial fraction coefficients $A_1, A_2 \dots A_n$ are found from the residues

$$A_i = \left[(s+a_i)\frac{N(s)}{D(s)}\right]_{s=-a_i}$$

as described in Appendix C. From the table of transforms in Table 2 each first-order term corresponds to an exponential time function,

$$e^{-at} \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{s+a},$$

so that the complete inverse transform is

$$f(t) = A_1 e^{-a_1 t} + A_2 e^{-a_2 t} + \dots + A_n e^{-a_n t}.$$

■ Example

Find the inverse Laplace transform of

$$F(s) = \frac{6s+14}{s^2+4s+3}.$$

Solution: The partial fraction expansion is

$$\begin{aligned} F(s) &= \frac{6s+14}{(s+3)(s+1)} \\ &= \frac{A_1}{s+3} + \frac{A_2}{s+1} \end{aligned}$$

where A_1, and A_2 are found from the residues

$$\begin{aligned} A_1 &= [(s+3)F(s)]_{s=-3} \\ &= \left[\frac{6s+14}{s+1}\right]_{s=-3} \\ &= 2, \end{aligned}$$

and similarly $A_2 = 4$. Then from Table 2,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} \\ &= 2e^{-3t} + 4e^{-t} \qquad \text{for} \quad t > 0. \end{aligned}$$

As described in Appendix C, if the denominator polynomial $D(s)$ contains repeated factors, the partial fraction expansion of $F(s)$ contains additional terms involving higher powers of the factor.

■ Example

Find the inverse Laplace transform of

$$F(s) = \frac{5s^2+3s+1}{s^3+s^2} = \frac{5s^2+3s+1}{s^2(s+1)}$$

Solution In this case there is a repeated factor s^2 in the denominator, and the partial fraction expansion contains an additional term:

$$\begin{aligned} F(s) &= \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s+1} \\ &= \frac{2}{s} + \frac{1}{s^2} + \frac{3}{s+1}. \end{aligned} \qquad (175)$$

The inverse transform of the three components can be found in Table 2, and the total solution is therefore

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 2 + t + 3e^{-t} \qquad \text{for} \quad t > 0. \end{aligned}$$

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7 Laplace Transform Applications in Linear Systems

7.1 Solution of Linear Differential Equations

The use of the derivative property of the Laplace transform generates a direct algebraic solution method for determining the response of a system described by a linear input/output differential equation. Consider an nth order linear system, completely relaxed at time $t = 0$, and described by

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u. \tag{176}$$

In addition assume that the input function $u(t)$, and all of its derivatives are zero at time $t = 0$, and that any discontinuities occur at time $t = 0^+$. Under these conditions the Laplace transforms of the derivatives of both the input and output simplify to

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s),$$

so that if the Laplace transform of both sides is taken

$$\left\{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right\} Y(s) = \left\{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right\} U(s) \tag{177}$$

which has had the effect of reducing the original differential equation into an algebraic equation in the complex variable s. This equation may be rewritten to define the Laplace transform of the output:

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} U(s) \tag{178}$$

$$= H(s) U(s) \tag{179}$$

The Laplace transform generalizes the definition of the transfer function to a complete input/output description of the system for any input $u(t)$ that has a Laplace transform.

The system response $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ may be found by decomposing the expression for $Y(s) = U(s)H(s)$ into a sum of recognizable components using the method of partial fractions as described above, and using tables of Laplace transform pairs, such as Table 2, to find the component time domain responses. To summarize, the Laplace transform method for determining the response of a system to an input $u(t)$ consists of the following steps:

(1) If the transfer function is not available it may be computed by taking the Laplace transform of the differential equation and solving the resulting algebraic equation for $Y(s)$.

(2) Take the Laplace transform of the input.

(3) Form the product $Y(s) = H(s)U(s)$.

(4) Find $y(t)$ by using the method of partial fractions to compute the inverse Laplace transform of $Y(s)$.

■ **Example**

Find the step response of first-order linear system with a differential equation

$$\tau \frac{dy}{dt} + y(t) = u(t)$$

Solution: It is assumed that the system is at rest at time $t = 0$. The Laplace transform of the unit step input is (Table 2):

$$\mathcal{L}\{u_s(t)\} = \frac{1}{s}. \tag{180}$$

Taking the Laplace transform of both sides of the differential equation generates

$$\begin{aligned} Y(s) &= \frac{1}{\tau s + 1} U(s) && (181) \\ &= \frac{1/\tau}{s(s + 1/\tau)} && (182) \end{aligned}$$

Using the method of partial fractions (Appendix C), the response may be written

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{1}{s + 1/\tau} && (183) \\ &= \mathcal{L}\{u_s(t)\} + \mathcal{L}\left\{e^{-t/\tau}\right\} && (184) \end{aligned}$$

from Table 2, and we conclude that

$$y(t) = 1 - e^{-t/\tau} \tag{185}$$

■ **Example**

Find the response of a second-order system with a transfer function

$$H(s) = \frac{2}{s^2 + 3s + 2}$$

to a one-sided ramp input $u(t) = 3t$ for $t > 0$.

Solution: From Table 2, the Laplace transform of the input is

$$U(s) = 3\mathcal{L}\{t\} = \frac{3}{s^2}. \tag{186}$$

Taking the Laplace transform of both sides

$$\begin{aligned} Y(s) = H(s)U(s) &= \frac{6}{s^2\left(s^2 + 3s + 2\right)} \\ &= \frac{6}{s^2\left(s + 2\right)\left(s + 1\right)} && (187) \end{aligned}$$

The method of partial fractions is used to break the expression for $Y(s)$ into low-order components, noting that in this case we have a repeated root in the denominator:

$$Y(s) = -\frac{9}{2}\left(\frac{1}{s}\right) + 3\left(\frac{1}{s^2}\right) + 6\left(\frac{1}{s+1}\right) - \frac{3}{2}\left(\frac{1}{s+2}\right) \quad (188)$$

$$= -\frac{9}{2}\mathcal{L}\{1\} + 3\mathcal{L}\{t\} + 6\mathcal{L}\left\{e^{-t}\right\} - \frac{3}{2}\mathcal{L}\left\{e^{-2t}\right\} \quad (189)$$

from the Laplace transforms in Table 2. The time-domain response is therefore

$$y(t) = -\frac{9}{2} + 3t + 6e^{-t} - \frac{3}{2}e^{-2t} \quad (190)$$

If the system initial conditions are not zero, the full definition of the Laplace transform of the derivative of a function defined in Eq. (173) must be used

$$\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s) - \sum_{i=1}^{n} s^{n-i}\left(\left.\frac{d^{i-1}y}{dt^{i-1}}\right|_{t=0}\right).$$

For example, consider a second-order differential equation describing a system with non-zero initial conditions $y(0)$ and $\dot{y}(0)$,

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b_1\frac{du}{dt} + b_0 u. \quad (191)$$

The complete Laplace transform of each term on both sides gives

$$a_2\left\{s^2Y(s) - sy(0) - y\prime(0)\right\} + a_1\{sY(s) - y(0)\} + a_0Y(s) = b_1 sU(s) + b_0U(s) \quad (192)$$

where as before it is assumed that a time $t = 0$ all derivatives of the input $u(t)$ are zero. Then

$$\left\{a_2s^2 + a_1s + a_0\right\}Y(s) = \{b_1s + b_0\}U(s) + c_1s + c_0 \quad (193)$$

where $c_1 = a_2(y(0) + \dot{y}(0))$ and $c_0 = a_1y(0)$. The Laplace transform of the output is the superposition of two terms; one a forced response due to $u(t)$, and the second a function of the initial conditions:

$$Y(s) = \frac{b_1s + b_0}{a_2s^2 + a_1s + a_0}U(s) + \frac{c_2s + c_0}{a_2s^2 + a_1s + a_0}. \quad (194)$$

The time-domain response also has two components

$$y(t) = \mathcal{L}^{-1}\left\{\frac{b_1s + b_0}{a_2s^2 + a_1s + a_0}U(s)\right\} + \mathcal{L}^{-1}\left\{\frac{c_1s + c_0}{a_2s^2 + a_1s + a_0}\right\} \quad (195)$$

each which may be found using the method of partial fractions.

■ Example

A mass $m = 18$ kg. is suspended on a spring of stiffness $K = 162$ N/m. At time $t = 0$ the mass is released from a height $y(0) = 0.1$ m above its rest position. Find the resulting unforced motion of the mass.

Solution: The system has a homogeneous differential equation

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \tag{196}$$

and initial conditions $y(0) = 0.1$ and $\dot{y}(0) = 0$. The Laplace transform of the differential equation is

$$\left\{s^2Y(s) - 0.1s\right\} + 9Y(s) = 0 \tag{197}$$

$$\left\{s^2 + 9\right\} Y(s) = 0.1s \tag{198}$$

so that

$$\begin{aligned} y(t) &= 0.1\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} && (199)\\ &= 0.1\cos 3t && (200)\end{aligned}$$

from Table 2.

7.2 Solution of State Equations

The Laplace transform solution method may be applied directly to a set of dynamic equations expressed in state-space form. Consider a linear system described by its state and output equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). && (201)\end{aligned}$$

and assume initially that the system is at rest at time $t = 0$, so that $\mathbf{x}(0) = \mathbf{0}$. Then taking the Laplace transform of both sides gives

$$\begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) && (202)\\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s). && (203)\end{aligned}$$

The state equations may be rearranged to solve explicitly for $X(s)$

$$\begin{aligned} [s\mathbf{I} - \mathbf{A}]\,\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) && (204)\\ \mathbf{X}(s) &= [s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) && (205)\end{aligned}$$

and substituted into the output equation

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)\\ &= \left(\mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s). && (206)\end{aligned}$$

The response of the system is the inverse Laplace transform of $\mathbf{Y}(s)$

$$\mathbf{y}(t) = \mathcal{L}^{-1}\left\{\left(\mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s)\right\} \tag{207}$$

For a single-input single-output system, the Laplace domain system response can be written

$$Y(s) = H(s)U(s)$$

where

$$H(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D} \tag{208}$$

is the system *transfer function*. Then

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$$

as before.

If the initial conditions on the state variables are not zero, so that the initial condition vector $\mathbf{x}(0) = \mathbf{x}_0$, the Laplace transform of the state equations must be modified to include the initial term in the Laplace transform of the derivative

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}_0 &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \left[s\mathbf{I} - \mathbf{A}\right]\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) + \mathbf{x}_0 \\ \mathbf{X}(s) &= \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B}\mathbf{U}(s) + \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}_0 \end{aligned} \tag{209}$$

The output equation then becomes

$$\mathbf{Y}(s) = \left\{\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}\right\}\mathbf{U}(s) + \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}_0. \tag{210}$$

which involves two terms, a forced component and an initial condition component. Then the time-domain response is the sum of the two inverse Laplace transforms

$$\mathbf{y}(t) = \mathcal{L}^{-1}\left\{\left(\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s)\right\} + \mathcal{L}^{-1}\left\{\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}_0\right\}. \tag{211}$$

7.3 The Convolution Property

The Laplace domain system representation has the same multiplicative input/output relationship as the Fourier transform domain. If a system input function $u(t)$ has both a Fourier transform and a Laplace transform

$$\begin{aligned} u(t) &\overset{\mathcal{F}}{\Longleftrightarrow} U(j\omega) \\ u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} U(s) \end{aligned}$$

then we have observed in Sections 5.2 and 7.1 that a multiplicative input/output relationship between system input and output exists in both the Fourier and Laplace domains

$$\begin{aligned} Y(j\omega) &= U(j\omega)H(j\omega) \\ Y(s) &= U(s)H(s). \end{aligned}$$

Since in the time domain the system response is defined by the convolution of the input and the system impulse response $h(t)$

$$y(t) = h(t) \star u(t)$$

the duality between the operations of convolution and multiplication therefore hold for the Laplace domain

$$h(t) \star u(t) \overset{\mathcal{L}}{\Longleftrightarrow} H(s)U(s). \tag{212}$$

As in the Fourier transform domain this property holds for any pair of functions

$$\mathcal{L}\left\{f(t) \star g(t)\right\} = F(s)G(s). \tag{213}$$

.

7.4 The Relationship between the Transfer Function and the Impulse Response

The impulse response $h(t)$ is defined as the system response to a Dirac delta function $\delta(t)$. Because the impulse has the property that its Laplace transform is unity, in the Laplace domain the transform of the impulse response is

$$h(t) = \mathcal{L}^{-1}\left\{H(s)U(s)\right\} = \mathcal{L}^{-1}\left\{H(s)\right\}.$$

In other words, the system impulse response and the transfer function form a Laplace transform pair

$$h(t) \overset{\mathcal{L}}{\Longleftrightarrow} H(s) \tag{214}$$

which is analogous to the Fourier transform relationship between the impulse response and the frequency response as shown in Section 5.3.

■ **Example**

Find the impulse response of a system with a transfer function

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Solution: The impulse response is the inverse Laplace transform of the transfer function $H(s)$:

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left\{H(s)\right\} & (215)\\ &= \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)}\right\} & (216)\\ &= \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\} & (217)\\ &= 2e^{-t} - 2e^{-2t} & (218)\end{aligned}$$

7.5 The Steady-State Response of a Linear System

The final value theorem, introduced in Section 6.2, states that if a time function has a steady-state value, then that value can be found from the limiting behavior of its Laplace transform as s tends to zero,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

This property can be applied directly to the response $y(t)$ of a system

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} sH(s)U(s) & (219)\end{aligned}$$

if $y(t)$ does come to a steady value as time t becomes large. In particular, if the input is a unit step function $u_s(t)$ then $U(s) = 1/s$, and the steady-state response is

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sH(s)\frac{1}{s} \\ &= \lim_{s\to 0} H(s) & (220)\end{aligned}$$

■ Example

Find the steady-state response of a system with a transfer function

$$H(s) = \frac{s+3}{(s+2)(s^2+3s+5)}$$

to a unit step input.

Solution: Using the final value theorem

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} [sH(s)U(s)] && (221)\\
&= \lim_{s\to 0} \left[sH(s)\frac{1}{s}\right] && (222)\\
&= \lim_{s\to 0} H(s) && (223)\\
&= \frac{3}{10} && (224)
\end{aligned}$$