

Simple Differential Equations - Introduction using Matlab

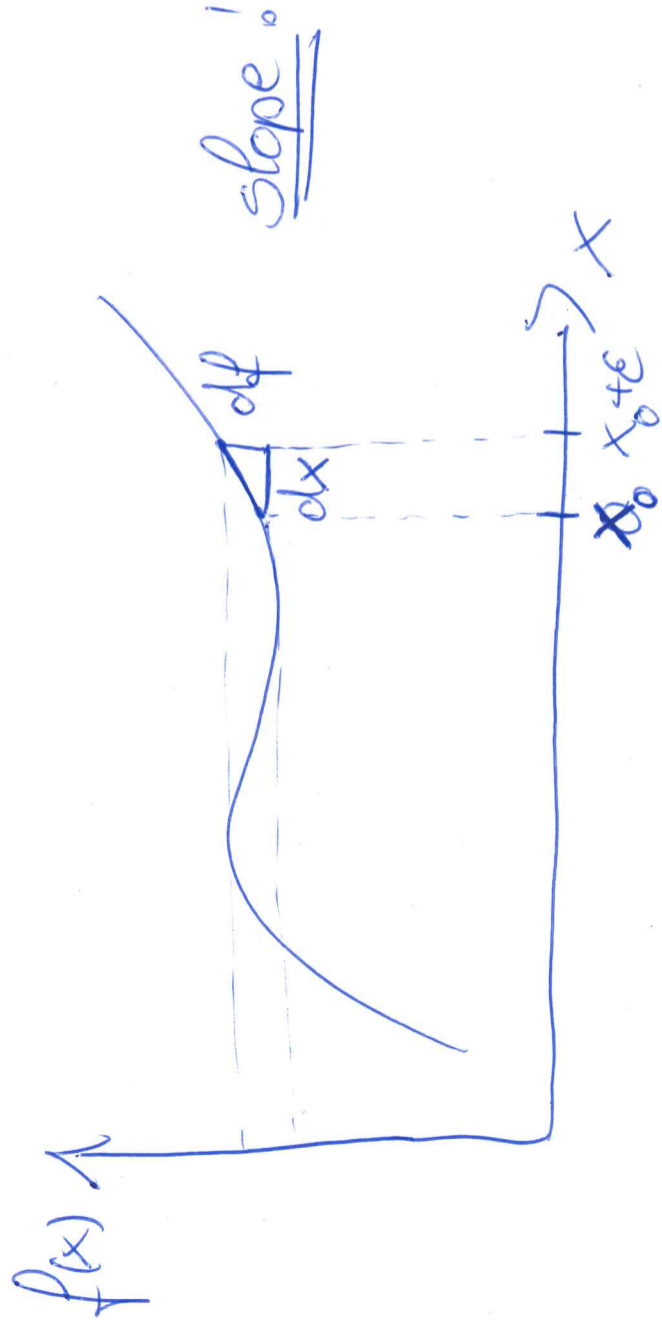
What is a derivative?

Let $f(x)$ be a function

$$f: x \in \mathbb{R} \rightarrow \mathbb{R}$$

derivative is defined by

$$\frac{d}{dx} f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$



notation:

$$\frac{d}{dx} f(x) = f'(x)$$

if x is time:

→ 1st derivative

- "slope"
- "velocity"

→ 2nd derivative

- "curvature"
- "acceleration"

$$\frac{d^n}{dx^n} f(x) = \frac{d}{dx} \left(\frac{d^{n-1}}{dx^{n-1}} f(x) \right)$$

another order: $\frac{d^0}{dx^0} f(x) = f(x)$

Simple rules:

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} c = 0$$

e.g. $\frac{d}{dx} x^2 = 2x$

$$\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

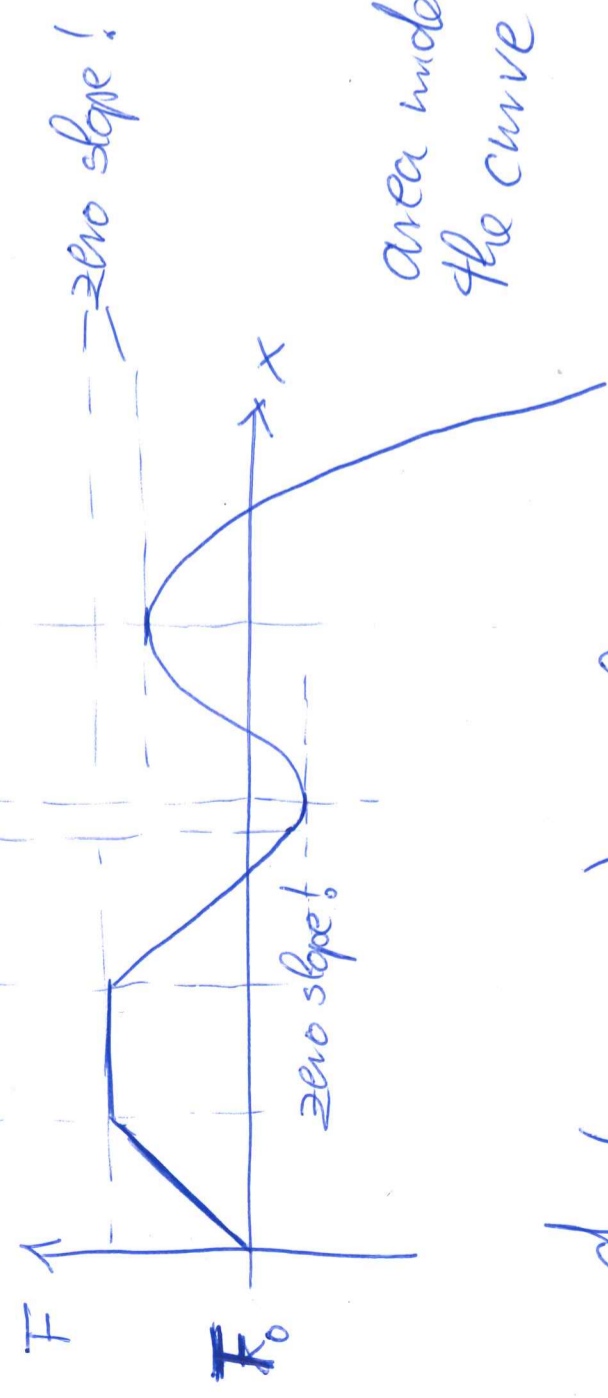
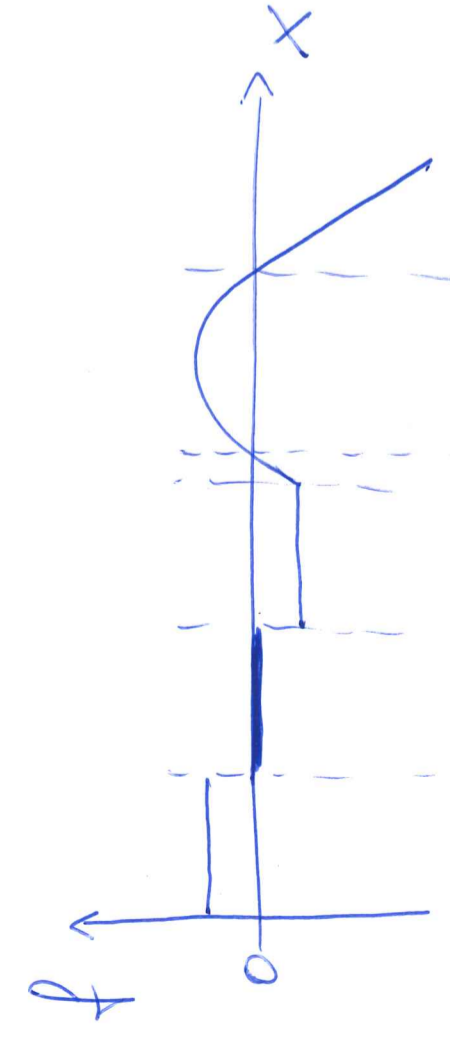
integration

Let $f(x)$ be a function

$$f(x) : x \in \mathbb{R} \rightarrow \mathbb{R}$$

Find $F(x)$ such that $F : x \in \mathbb{R} \rightarrow \mathbb{R}$

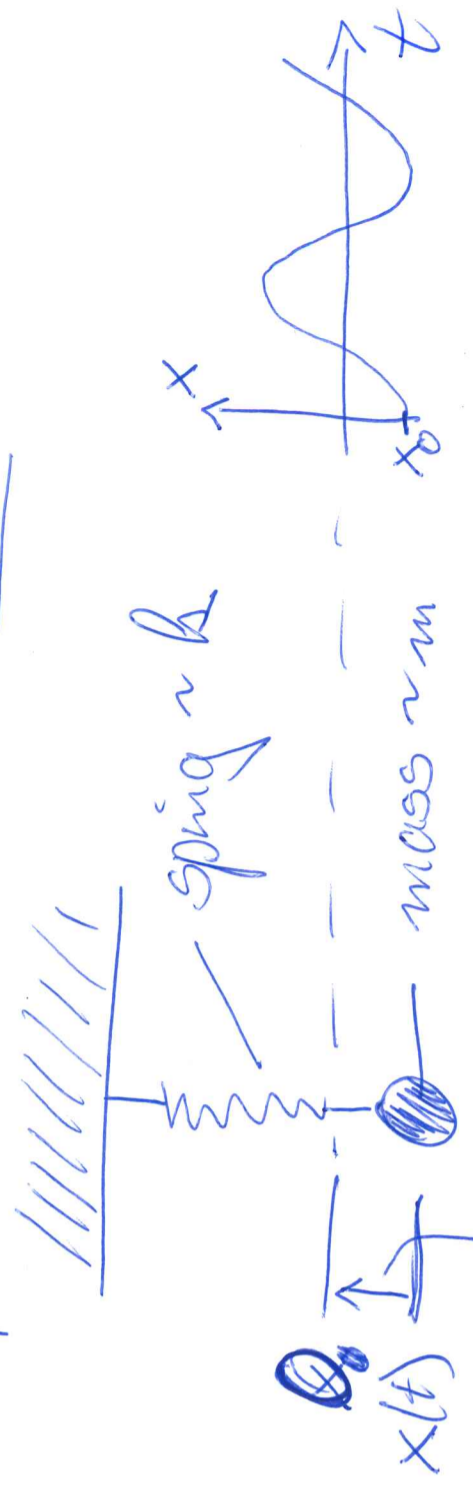
$$\frac{d}{dx} F(x) = f(x)$$



$$\frac{d}{dx} (F(x) + C) = f(x)$$

Why differential equations?

Example: harmonic oscillator



$$\text{force } F(t) = -kx(t)$$

$$\text{acceleration } \ddot{x}(t) = \frac{F(t)}{m}$$

$$m \ddot{x} = -kx$$

$$m \ddot{x} + kx = 0$$

Solution:

$$\omega_0^2 = \frac{k}{m}$$

$$x(t) = x_0 \cos \omega_0 t$$

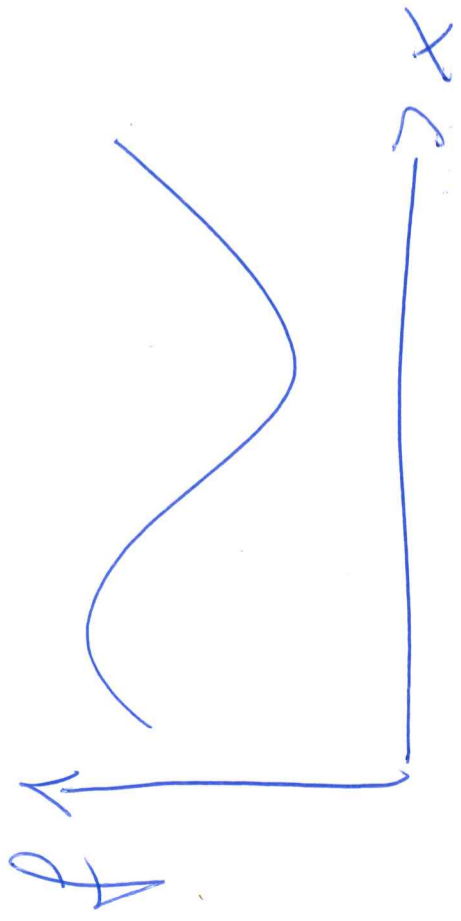
$$\dot{x}(t) = -x_0 \omega_0 \sin \omega_0 t$$

$$\ddot{x}(t) = -x_0 \omega_0^2 \cos \omega_0 t$$

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Let $x \in \mathbb{R}$ be a variable

and $f: x \in \mathbb{R} \rightarrow \mathbb{R}$
be a function



Ordinary Differential Equations (ODE):

$$\frac{d^n}{dx^n} f(x) = G \left(\frac{d^{n-1}}{dx^{n-1}} f, \frac{d^{n-2}}{dx^{n-2}} f, \dots, \frac{d}{dx} f, x \right)$$

order n
 general function
 $G: \mathbb{R} \rightarrow \mathbb{R}$

initial and boundary values

The differential equation does not specify the solution uniquely!

Example:

$x(t) = x_0 \cdot \cos \omega_0 t$ is a solution of the harmonic oscillator

but also

$x'(t) = x_0 \cdot \cos(\omega_0 t + \alpha)$ is a solution for all $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$

one has to specify boundary conditions

E.g. $x(t=0) = x_0$ initial ~~value~~ ^{coordinate}
 $\frac{dx}{dt} \Big|_{t=0} = v_0$ initial velocity

or boundary values eg. catenary curve
 $\cosh(x)$

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higher order ODEs can be rewritten as system of coupled 1st order ODEs:

define:

$$\begin{aligned}
 y_1 &= \frac{d}{dx} f(x) \\
 y_2 &= \frac{d^2}{dx^2} f(x) = \frac{d}{dx} y_1(x) \\
 &\vdots \\
 y_n &= \frac{d^n}{dx^n} f(x) = \frac{d}{dx} y_{n-1}(x) \\
 &= G(y_{n-1}, y_{n-2}, \dots, y_1, x)
 \end{aligned}$$

if G is a linear function of all arguments the ODE can be written in matrix form

$$\frac{d}{dx} \vec{y} = A \vec{y}$$

see lecture on matrix calculus 2017/09/29

solution: $\vec{y}(x) = e^{Ax} \vec{y}_0 \mid \vec{y}(0) = \vec{y}_0$

Example: damped harmonic oscillator

$$\frac{d^2}{dx^2} f(x) + \gamma \frac{d}{dx} f(x) + \text{sign}(c) \omega_0^2 f(x) = \text{driving force}$$

if $P \equiv 0$ "autonomous"

\cup attraction for $C > 0$
 \cap repulsion for $C < 0$

Let $f_{ext} \equiv 0$ and define $g := \frac{d}{dx} f(x)$ then

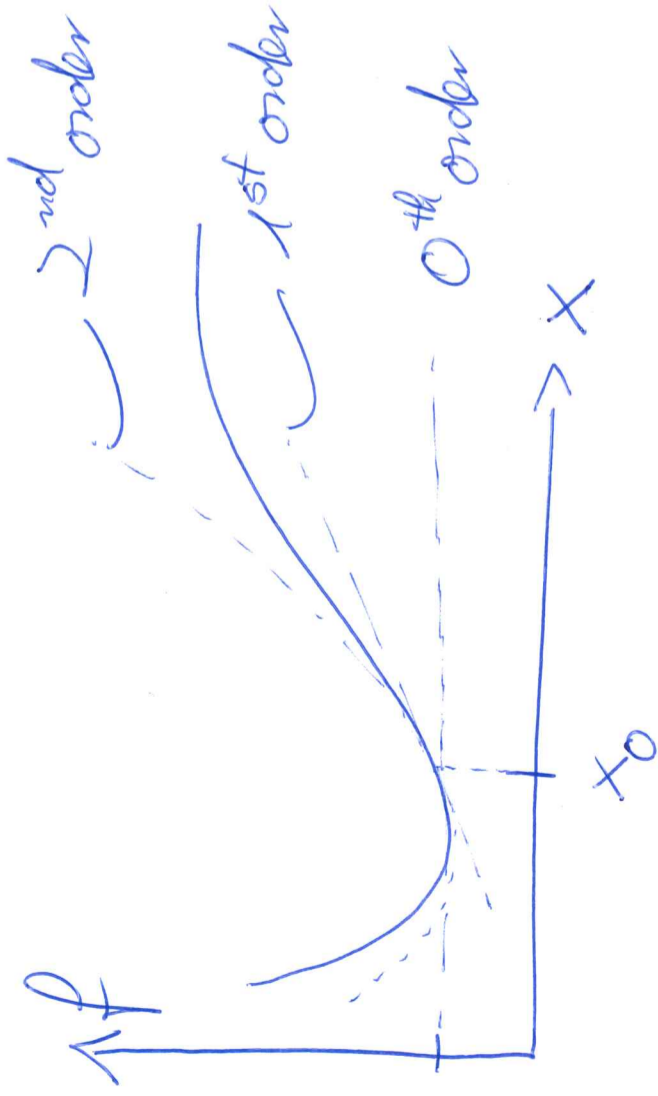
$$\frac{d}{dx} \begin{pmatrix} f \\ g \end{pmatrix} + \gamma \begin{pmatrix} g \\ f \end{pmatrix} + \text{sign}(c) \omega_0^2 \begin{pmatrix} f \\ 0 \end{pmatrix} = 0$$

$$= \frac{d}{dx} \begin{pmatrix} f \\ g \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\text{sign}(c)\omega_0^2 & -\gamma \end{pmatrix}}_{=: A} \begin{pmatrix} f \\ g \end{pmatrix}$$

exact solution!

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Taylor expansion of a function



$$f(x) = f(x_0) + (x-x_0) \cdot \left. \frac{df}{dx} \right|_{x_0} + \frac{1}{2}(x-x_0)^2 \left. \frac{d^2f}{dx^2} \right|_{x_0} + \dots$$

general:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (x-x_0)^n \left. \frac{d^n f}{dx^n} \right|_{x_0}$$

polynomial expansion of a general function $f(x)$ around an expansion point x_0

Numerical solution of ODEs

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if coupled ODEs: scalar \rightarrow vector

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

ODE:

$$\frac{d}{dx} f(x) = G(f(x), x)$$

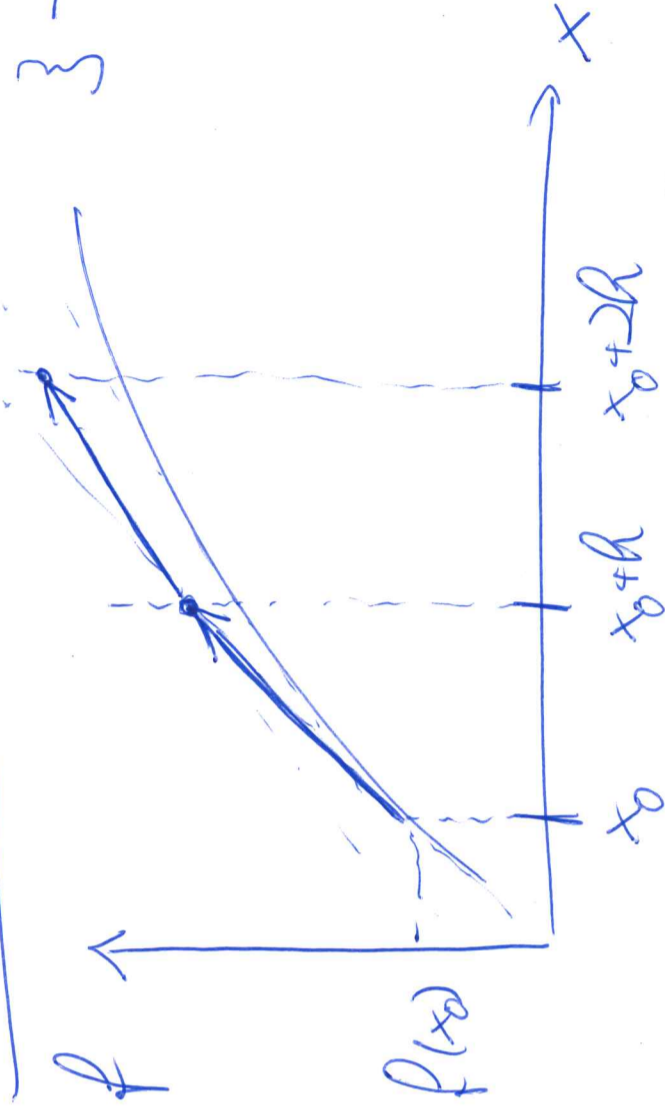
Taylor expansion of $f(x)$ around present state x_0 to 1st order:

$$f(x_0+h) = f(x_0) + \left. \frac{df}{dx} \right|_{x_0} \cdot h + \mathcal{O}(h^2)$$

known! rhs of ODE

$$= G(f(x_0), x_0)$$

Euler method:



$\} \rightarrow$ error!