

Matrix calculus - intro using Matlab

↳ not a linear algebra class!

Matlab = Matrix laboratory

→ highly optimized for matrix mult.

→ vectorize code whenever possible

↳ performance gain → example

↳ more compact

↳ easier to maintain

1) What is a vector? "element of a vector space"

→ ordered array of numbers

↳ rules for addition

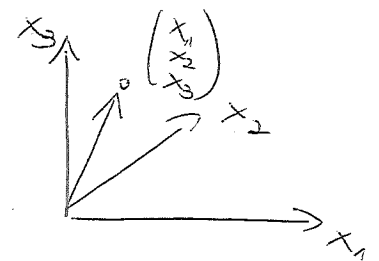
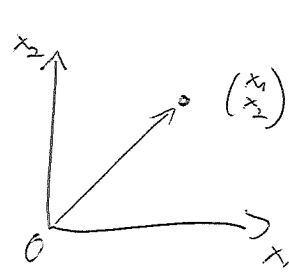
↳ rules for multiplication

column vector: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

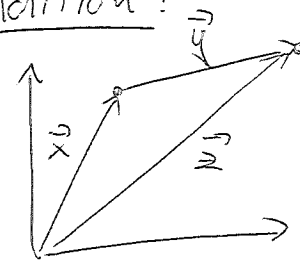
row vector: $\vec{y} = (y_1, y_2, \dots, y_n)$

ordered: $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ -2

visualization in 2D and 3D



addition:



$$\vec{z} = \vec{x} + \vec{y} \in \mathbb{R}^n$$

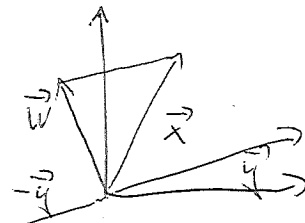
$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

negative: vector that cancels the addition of \vec{x}

$$\vec{x} + (-\vec{x}) = \vec{0}$$

$$-\vec{x} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$$

subtraction:



$$\vec{w} = \vec{x} + (-\vec{y}) = \vec{x} - \vec{y}$$

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{pmatrix}$$

scalar product = dot product = inner prod. 3-

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i = r$$

$$\vec{x}, \vec{y} \in \mathbb{R}^n \quad r \in \mathbb{R}$$

vectors scalar!

norm of a vector, length of a vector

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^2}$$
$$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \in \mathbb{R}$$

properties:

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{"commutes"}$$

$$a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$$

$$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$$

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} \quad \text{commutes}$$

special vectors:

-4

zero vector $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

unit vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$$

one vector \neq unit vector

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \neq \vec{e}_i \quad i = 1, \dots, n$$

2) What is a matrix?

-5-
Start level

again an ordered array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & & & & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

1st index: row 1...n

2nd index: column 1...m

row vector of column vectors: $(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m)$

$$\vec{a}_i \in \mathbb{R}^n \quad \forall i=1 \dots m$$

column vector of row vectors:

$$\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix} \quad \vec{a}^j \in \mathbb{R}^m \quad \forall j=1 \dots n$$

vectors are special cases of matrices!

short-hand notation

$$A = (a_{ij}) \quad \begin{matrix} i=1 \dots n \\ j=1 \dots m \end{matrix}$$

addition + subtraction:

$$A + B = (a_{ij} + b_{ij}) \in \mathbb{R}^{n \times m}$$

$$A - B = (a_{ij} - b_{ij})$$

$$-A = (-a_{ij})$$

multiplication:

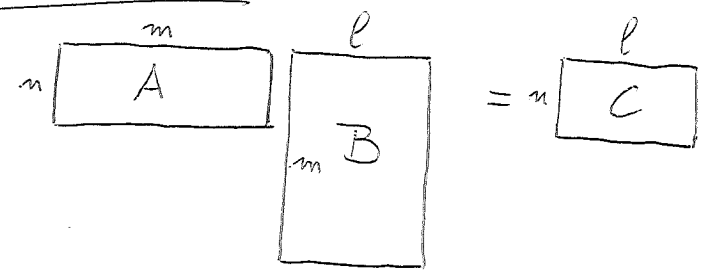
$$A \in \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}^{m \times l}$$

$$C = A \cdot B \in \mathbb{R}^{n \times l}$$

attention: we can't multiply any matrices
inner dimension ^m must be equal!

$$C = A \cdot B = (c_{ij}) = \left(\sum_{k=1}^m a_{ik} \cdot b_{kj} \right)$$

Symbolization:



row i of A ($i=1 \dots n$) is multiplied
with column j of B ($j=1 \dots l$)

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$ -7-

then $C = A \cdot B \in \mathbb{R}^{n \times n}$

and $D = B \cdot A \in \mathbb{R}^{m \times m}$

$$\boxed{A} \boxed{B} = \boxed{C} \quad | \quad \boxed{B} \boxed{A} = \boxed{D}$$

in general $A \cdot B \neq B \cdot A$!

transposition

$$A = (a_{ij}) \rightarrow A^T = (a_{ji})$$

interchange rows and columns
 refers to dot product
 $r = \sum_j a_{rj} a_{sj}$ and dyadic prod
 $R = \sum_j \square = \square$

$$\begin{aligned} (A^T)^T &= A \\ (A \cdot B)^T &= B^T \cdot A^T \end{aligned}$$

special matrices:

zero matrix $O_n = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$

$$A + O = O + A = A$$

unit matrix = identity matrix $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & & 1 \end{pmatrix}$

$$A \cdot I_n = I_n \cdot A = A$$

one-matrix \neq unit matrix -8-

$$\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

symmetric matrix $A^T = A$

determinant of a matrix

scalar expressing key properties of a square matrix

recursive definition:

$$n=1: \det A^{1 \times 1} = a_{11}$$

$$n>1: \det A^{n \times n} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_i$$

where $A_i \in \mathbb{R}^{(n-1) \times (n-1)}$
 is A after deletion of row i
 and column j

$$A_{ij} = \begin{pmatrix} \uparrow & & \\ & A & \\ & & \downarrow \end{pmatrix}_i$$

special case $n=2$:

$$\det A^{2 \times 2} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

A matrix is \rightarrow "singular" if $\det A = 0$ (X)
 \searrow "nonsingular" if $\det A \neq 0$ (+)

rules:

$$\det \mathbb{1}_n = 1$$

$$\det A^T = \det A$$

$$\det A \cdot B = \det A \cdot \det B$$

$$\det \text{diag } d_{ii} = \prod_{i=1}^n d_{ii}$$

$$\det (a \cdot A) = a^n \det A$$

trace of a square matrix

another scalar characterizing a matrix

$$\text{tr } A^{n \times n} = \text{Sp } A^{n \times n} = \sum_{i=1}^n a_{ii}$$

inverse of a square matrix

nonsingular matrices have a unique inverse:

↳ what's that?

$$A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{1}_n$$

$$A \in \mathbb{R}^n \\ \det A \neq 0$$

rules:

$$\det A^{-1} = \frac{1}{\det A}$$

prove this!

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(A^{-1})^T = (A^T)^{-1}$$

-9-

Systems of linear equations

-10

$$(1) \quad a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1m} \cdot x_m = b_1$$

$$(2) \quad a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2m} \cdot x_m = b_2$$

⋮

$$(n) \quad a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \dots + a_{nm} \cdot x_m = b_n$$

rewrite this as matrix products:

$$A \cdot \vec{x} = \vec{b} \quad \text{with} \quad A \in \mathbb{R}^{n \times m} \\ \vec{x} \in \mathbb{R}^m$$

(n × m)-matrices transform $\vec{b} \in \mathbb{R}^n$

m-dim. vectors into n-dim. vectors

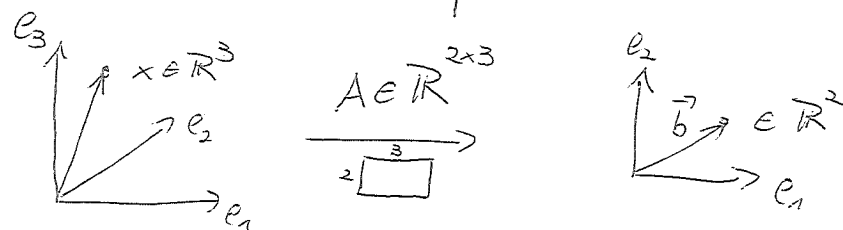
solve this equation = find solutions \vec{x}

for given A and \vec{b}

special case of square matrices:

$$\text{if } A^{-1} \text{ exists: } \vec{x} = A^{-1} \cdot A \vec{x} = A^{-1} \vec{b}$$

calculate determinant first!



eigenvector equations

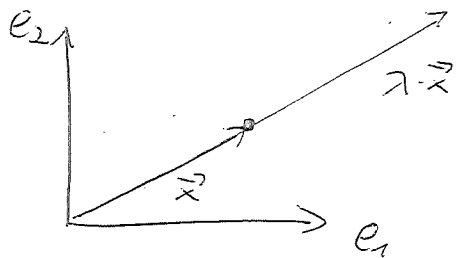
-11-

$$A \cdot \vec{x} = \lambda \vec{x}$$

$$A \in \mathbb{R}^{n \times n}$$

$$\vec{x} \in \mathbb{R}^n$$

$$\lambda \in \mathbb{R}$$



action of A on \vec{x}
is a scaling only!

solve system of equations

$$(A - \lambda I) \vec{x} = \vec{0}$$

orthogonal matrices

$$A^T \cdot A = A \cdot A^T = I_n \iff A^T = A^{-1}$$

orthogonal vectors via dot product:

$$\vec{x}_i \cdot \vec{x}_j = 0$$

$$\vec{x}_i \cdot \vec{x}_i = 1$$

$$\forall i \neq j$$

"normalization"