BEMESCO Lecture 2017/11/24

## Simple Differential Equations – Introduction using Matlab

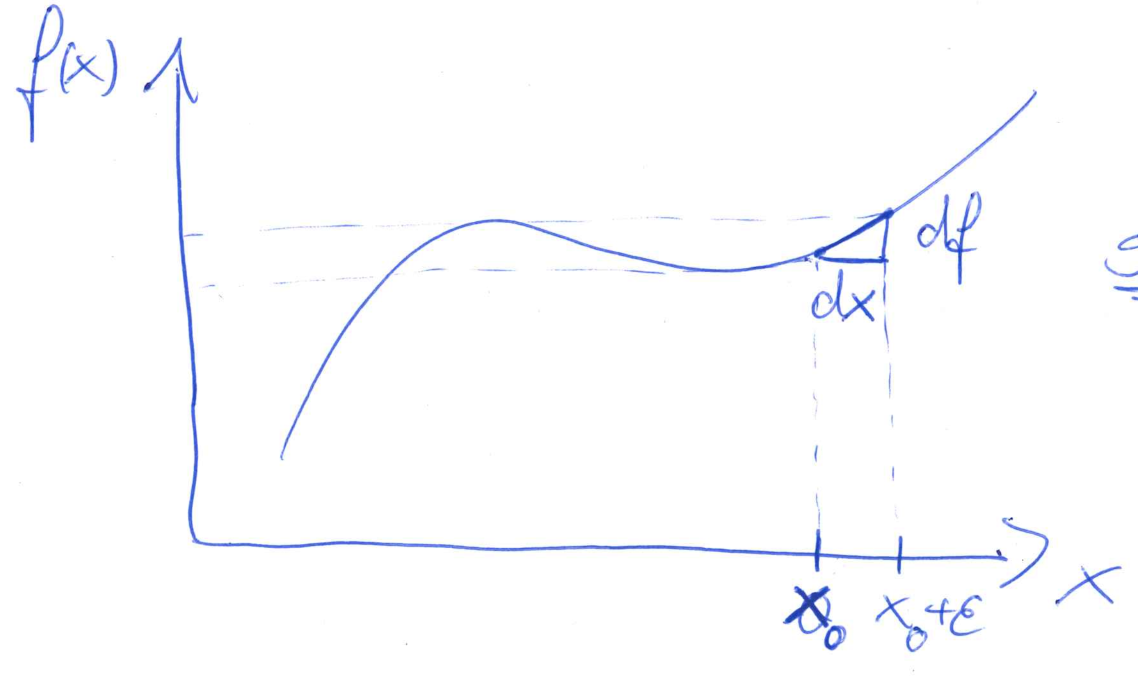
## What is a derivative ?

Let $f(x)$ be a function

$$f : x \in \mathbb{R} \longrightarrow \mathbb{R}$$

derivative is defined by

$$\frac{d}{dx} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$




slope !



notation :

$$\frac{d}{dx} f(x) = f'(x)$$

if $x$ is time :

$$\frac{d}{dt} f(t) = \dot{f}(t)$$

→ 1st derivative
- „slope",
- „velocity"

→ 2nd derivative
- „curvature"
- „acceleration"

higher derivatives :

$$\frac{d^n}{dx^n} f(x) = \frac{d}{dx} \left( \frac{d^{n-1}}{dx^{n-1}} f(x) \right)$$

zeroth order : $\frac{d^0}{dx^0} f(x) = f(x)$

simple rules :

$$\frac{d}{dx} x^n = n x^{n-1} \qquad \frac{d}{dx} c = 0$$

$$\text{e.g. } \frac{d}{dx} x^2 = 2x \qquad \frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx} e^{ax} = a e^{ax} \qquad \frac{d}{dx} \ln x = \frac{1}{x}$$

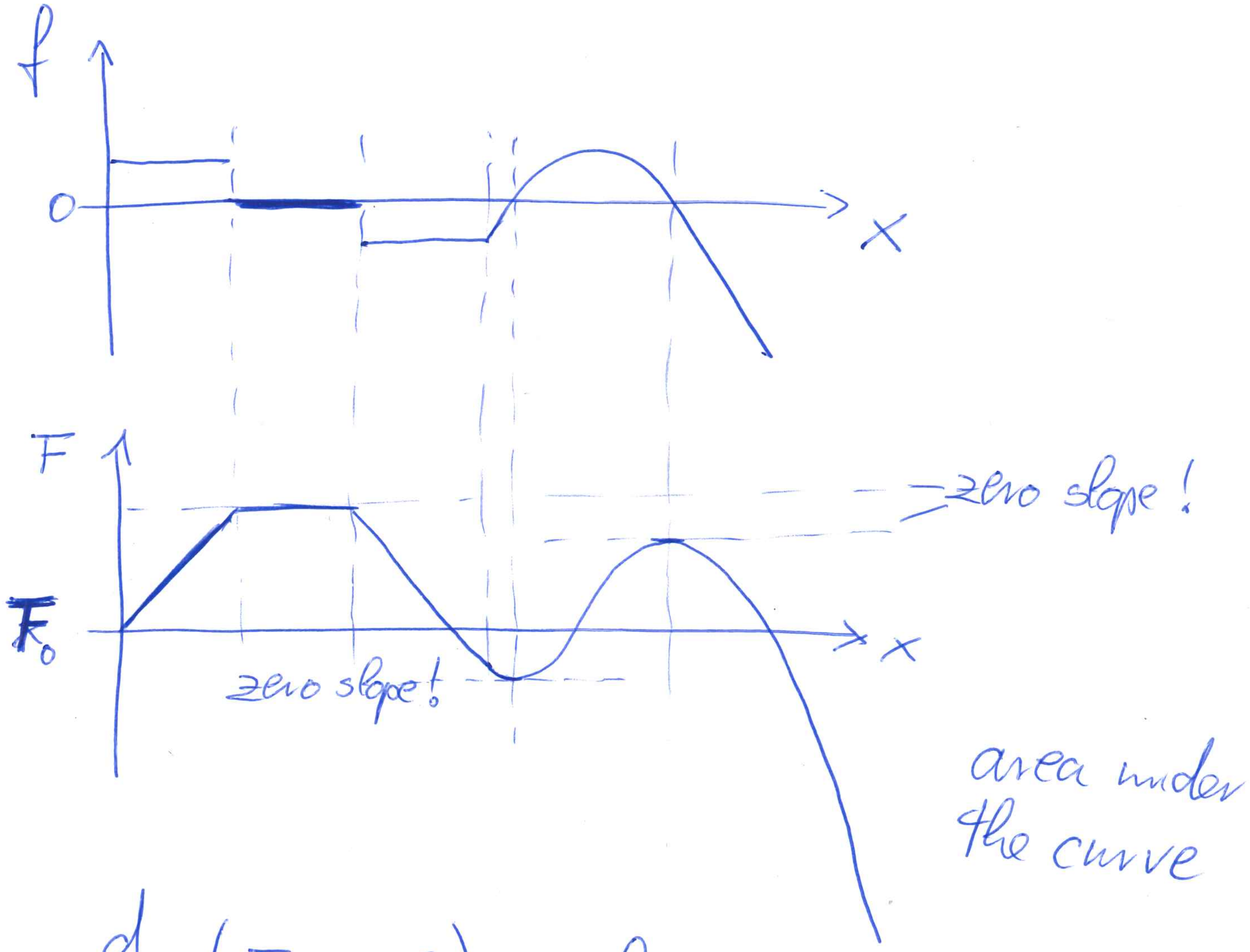
## integration

Let $f(x)$ be a function

$$f(x) : x \in \mathbb{R} \to \mathbb{R}$$

Find $F(x)$ such that $F : x \in \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dx} F(x) = f(x)$$

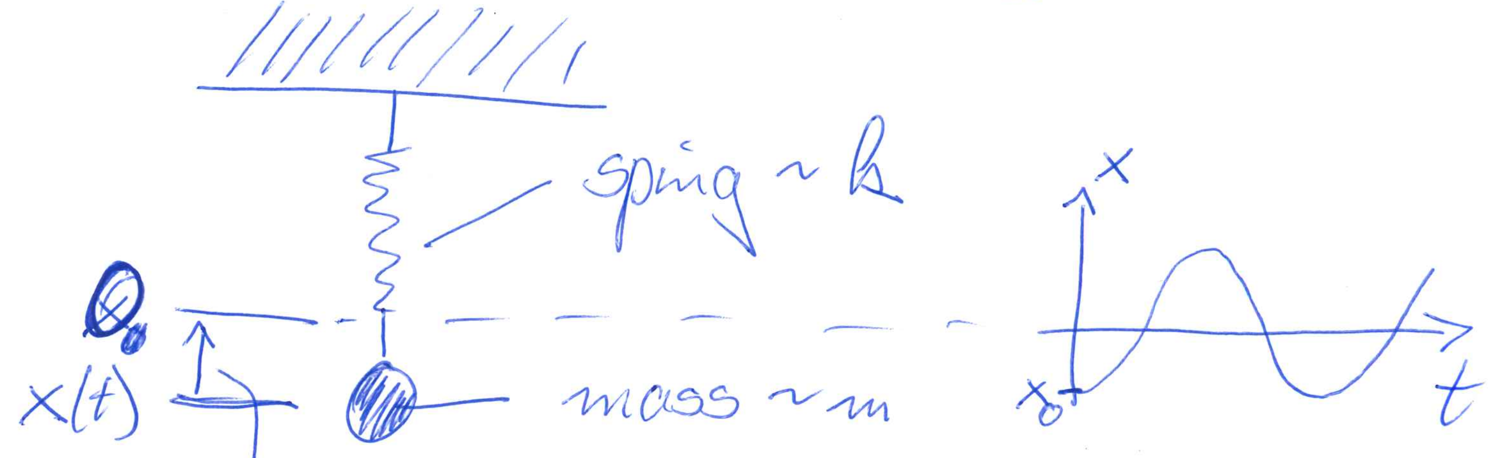



$$\frac{d}{dx}\left(F(x) + C\right) = f(x)$$



## Why differential equations?

**example : harmonic oscillator**




force $F(t) = -kx(t)$

acceleration $\ddot{x}(t) = \frac{F(t)}{m}$

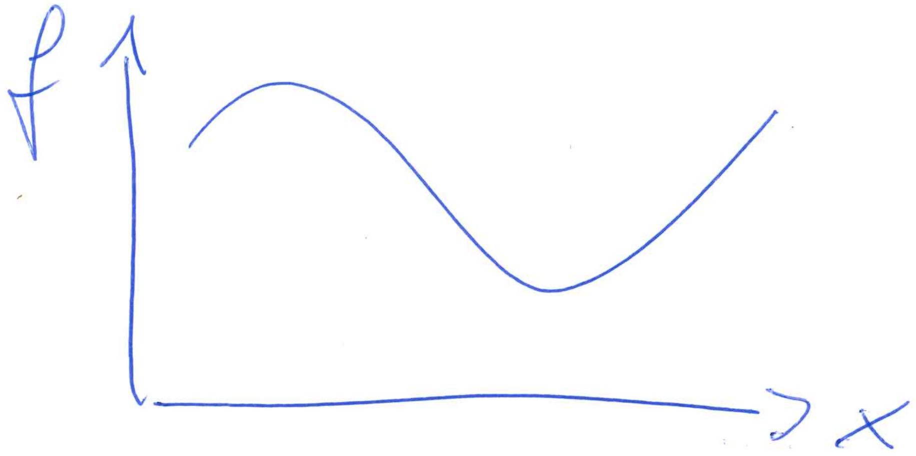
$$m\ddot{x} = -kx$$
$$m\ddot{x} + kx = 0$$

solution :

$$x(t) = x_0 \cos \omega_0 t \qquad \omega_0^2 = \frac{k}{m}$$
$$\dot{x}(t) = x_0 \omega_0 \sin \omega_0 t$$
$$\ddot{x}(t) = -x_0 \omega_0^2 \cos \omega_0 t$$

Let $x \in \mathbb{R}$ be a variable

and
$$f : x \in \mathbb{R} \to \mathbb{R}$$
be a function




## Ordinary Differential Equations (ODE):

$$\frac{d^n}{dx^n} f(x) = G\left(\frac{d^{n-1}}{dx^{n-1}}f, \frac{d^{n-2}}{dx^{n-2}}f, \ldots, \frac{d}{dx}f, x\right)$$

order $n$

general function

$$G : \mathbb{R} \to \mathbb{R}$$



## Initial and boundary values

The differential equation does <u>not</u> specify the solution uniquely!

Example:

$x(t) = x_0 \cdot \cos \omega_0 t$ is a solution of the harmonic oscillator

but also

$x'(t) = x_0 \cdot \cos(\omega_0 t + \alpha)$ is a solution for all $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$

one has to specify <u>boundary conditions</u>

e.g. $x(t=0) = x_0$ <u>initial</u> ~~value~~ coordinate

$\left.\frac{dx}{dt}\right|_{t=0} = v_0$ <u>initial</u> velocity

or <u>boundary values</u> eg. catenary curve

$\cosh(x)$

higher order ODEs can be rewritten as system of <u>coupled 1st order ODEs</u>:

define:

$$y_1 = \frac{d}{dx} f(x)$$

$$y_2 = \frac{d^2}{dx^2} f(x) = \frac{d}{dx} y_1(x)$$

$$\vdots$$

$$y_n = \frac{d^n}{dx^n} f(x) = \frac{d}{dx} y_{n-1}(x)$$

$$= G\left(y_{n-1}, y_{n-2}, \dots, y_1, x\right)$$

if $G$ is a <u>linear function</u> of all arguments the ODE can be written in <u>matrix form</u>

$$\frac{d}{dx} \vec{y} = A \vec{y}$$

see lecture on <u>matrix calculus</u> 2017/09/29

solution: $\vec{y}(x) = e^{Ax} \vec{y}_0 \;\Big|\; \vec{y}(0) = \vec{y}_0$



<u>Example</u>:

<u>damped harmonic oscillator</u>

$$\frac{d^2}{dx^2} f(x) + \gamma \frac{d}{dx} f(x) + \mathrm{sign}(c)\,\omega_0^2 f(x) = f_{ext}(x)$$

∪ attraction for $c > 0$

∩ repulsion for $c < 0$

driving force

if $\equiv 0$ „<u>autonomous</u>"

let $f_{ext} \equiv 0$ and define $g := \frac{d}{dx} f(x)$

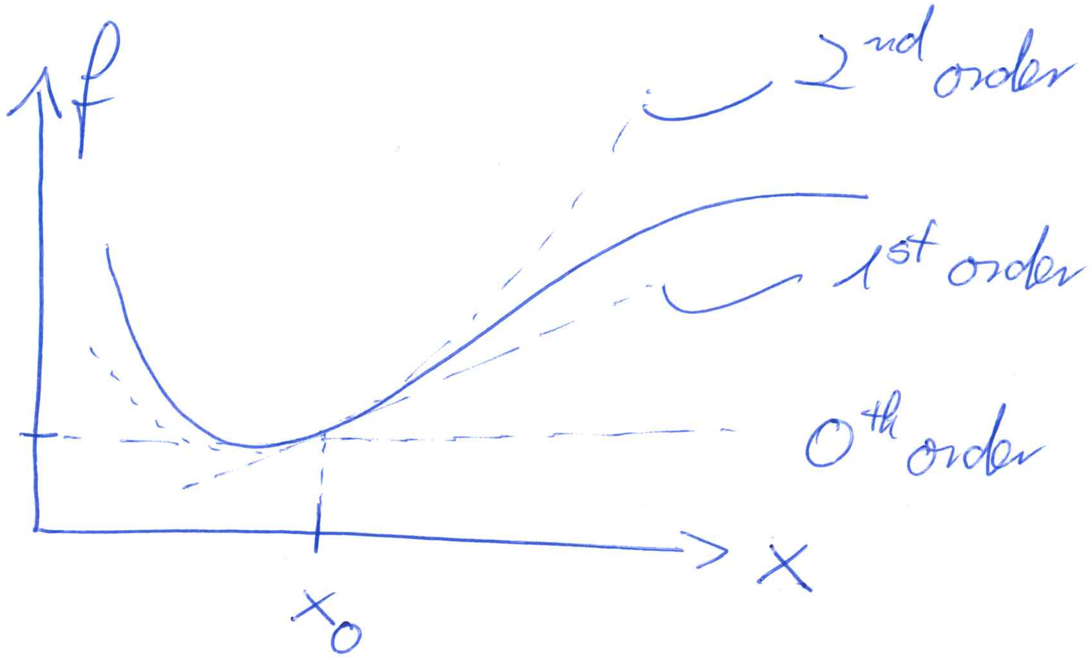
then

$$\frac{d}{dx} g(x) + \gamma \underbrace{g(x)}_{= \frac{d}{dx} f(x)} + \mathrm{sign}(c)\,\omega_0^2 f(x) = 0$$

$$\frac{d}{dx} \begin{pmatrix} f \\ g \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\mathrm{sign}(c)\,\omega_0^2 & -\gamma \end{pmatrix}}_{=: A} \begin{pmatrix} f \\ g \end{pmatrix}$$

<u>exact</u> solution!

## Taylor expansion of a function




$$f(x) = f(x_0) + (x-x_0)\cdot \frac{df}{dx}\Big|_{x_0} + \frac{1}{2}(x-x_0)^2 \frac{d^2f}{dx^2}\Big|_{x_0}$$
$$+ \dots$$

general:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-x_0)^n \frac{d^nf}{dx^n}\Big|_{x_0}$$

polynomial expansion of a general function $f(x)$ around an expansion point $x_0$



## Numerical solution of ODEs

if coupled ODEs: scalar $\to$ vector

$f \qquad \vec{f} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

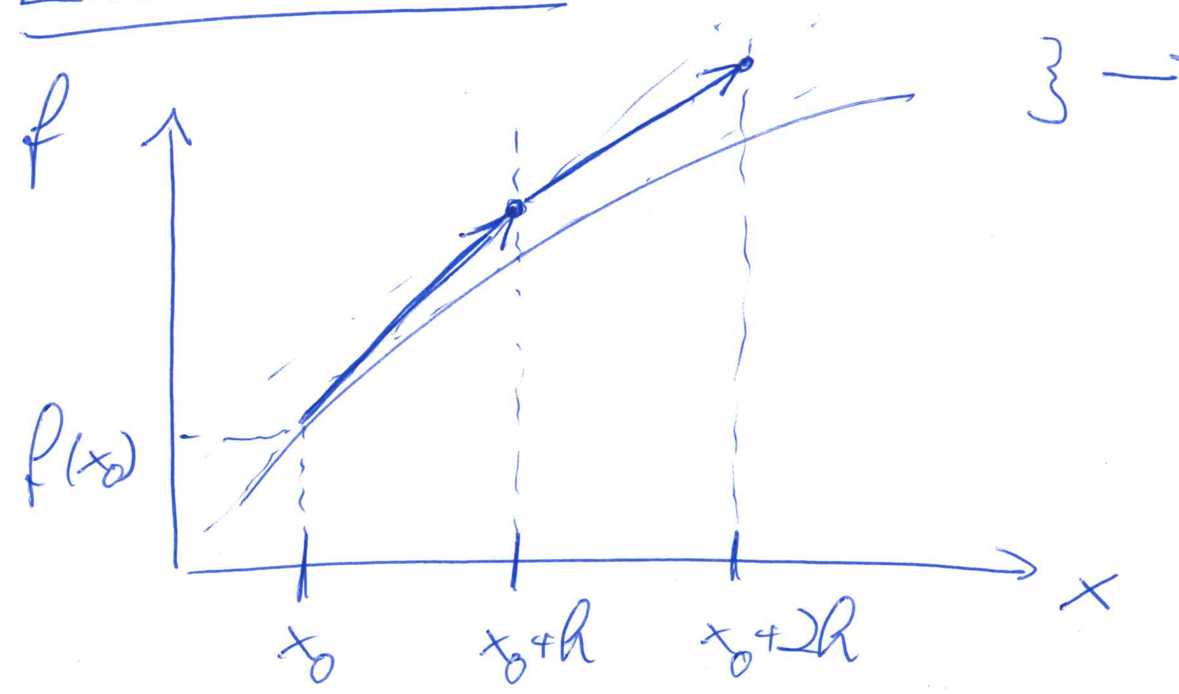
ODE:

$$\frac{d}{dx} f(x) = G(f(x), x)$$

Taylor expansion of $f(x)$ around present state $x_0$ to 1st order:

$$f(x_0 + h) = f(x_0) + \underbrace{\frac{df}{dx}\Big|_{x_0}}_{\text{known! rhs of ODE} = G(f(x_0), x_0)} \cdot h + \mathcal{O}(h^2)$$

Euler method:




$\} \to$ error!