BENESCO Lecture 2017/09/29

Matrix calculus – intro using Matlab

↳ not a linear algebra class!

Matlab = Matrix laboratory

→ highly optimized for matrix mult.

→ vectorize code whenever possible

↳ performance gain → example

↳ more compact

↳ easier to maintain

1) What is a vector?

"element of a vector space"

→ ordered array of numbers

↳ rules for addition

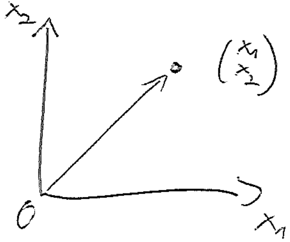
↳ rules for multiplication

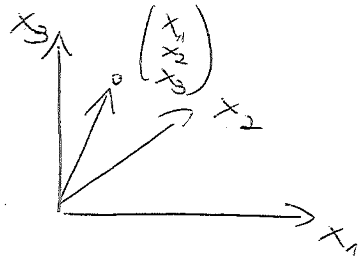
column vector: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

row vector: $\vec{y} = (y_1, y_2 \cdots y_n)$

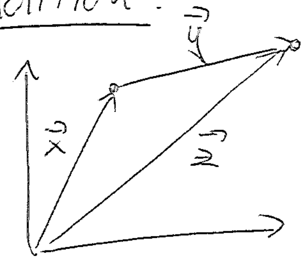
ordered: $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

visualization in 2D and 3D





addition:



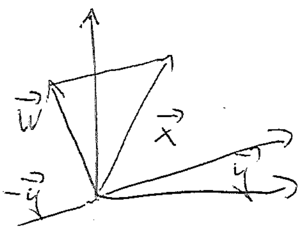
$\vec{z} = \vec{x} + \vec{y} \in \mathbb{R}^n$

$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

negative: vector that cancels the addition of $\vec{x}$

$\vec{x} + (-\vec{x}) = \vec{0}$ $\qquad -\vec{x} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$

subtraction:



$\vec{w} = \vec{x} + (-\vec{y}) = \vec{x} - \vec{y}$

$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{pmatrix}$

scalar product = dot product = inner prod.

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = r$$

$\vec{x}, \vec{y} \in \mathbb{R}^n$ vectors $\qquad r \in \mathbb{R}$ scalar!

norm of a vector, length of a vector

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^2}$$
$$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \in \mathbb{R}$$

properties:

$\vec{x} + \vec{y} = \vec{y} + \vec{x}$ „commutes"

$a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ commutes

special vectors:

zero vector $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

$$\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

unit vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$$

one vector $\neq$ unit vector

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \neq \vec{e}_i \qquad i = 1 \dots n$$

2) What is a matrix?

Start here!

again an ordered array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ a_{n1} & \cdots & \cdots & & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

1st index : row $1 \dots n$

2nd index : column $1 \dots m$

row vector of column vectors :

$$(\vec{a}_1, \vec{a}_2, \dots \vec{a}_m)$$

$$\vec{a}_i \in \mathbb{R}^n \qquad \forall\, i = 1 \dots m$$

column vector of row vectors :

$$\begin{pmatrix} \vec{a}^1 \\ \vec{a}^2 \\ \vdots \\ \vec{a}^n \end{pmatrix} \qquad \vec{a}^j \in \mathbb{R}^m \qquad \forall\, j = 1 \dots n$$

vectors are special cases of matrices!

short-hand notation

$$A = (a_{ij}) \qquad \begin{matrix} i = 1 \dots n \\ j = 1 \dots m \end{matrix}$$

addition + subtraction :

$$A + B = (a_{ij} + b_{ij}) \qquad \in \mathbb{R}^{n \times m}$$

$$A - B = (a_{ij} - b_{ij})$$

$$-A = (-a_{ij})$$

multiplication :

$$A \in \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}^{m \times l}$$

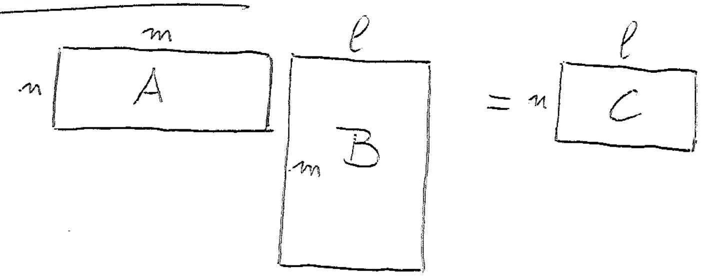
$$C = A \cdot B \quad \in \mathbb{R}^{n \times l}$$

attention : we can't multiply any matrices

inner dimension m must be equal!

$$C = A \cdot B = (c_{ij}) = \left(\sum_{k=1}^{m} a_{ik} \cdot b_{kj} \right)$$

Symbolization :



row i of A $(i = 1 \dots n)$ is multiplied with column j of B $(j = 1 \dots l)$

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$

then $C = A\cdot B \in \mathbb{R}^{n\times n}$

and $D = B\cdot A \in \mathbb{R}^{m\times m}$

$$\boxed{\;A\;}\,\boxed{B} = \boxed{C} \qquad \Big| \qquad \boxed{B}\,\boxed{\;A\;} = \boxed{\;D\;}$$

in general $A\cdot B \neq B\cdot A$!

transposition

$$A = (a_{ij}) \longrightarrow A^T = (a_{ji})$$

$(A^T)^T = A$

$(A\cdot B)^T = B^T A^T$

interchange rows and columns

refer to dot product $r = \vec{x}^T \vec{y}$ and dyadic prod R

special matrices:

zero matrix $O_n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

$$A + O = O + A = A$$

unit matrix = identity matrix $\mathbb{1}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

$$A^{n\times n}\cdot \mathbb{1}_n = \mathbb{1}_n \cdot A^{n\times n} = A$$

one-matrix $\neq$ unit matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

symmetric matrix $A^T = A$

determinant of a matrix

scalar expressing key properties of a square matrix

recursive definition:

$n = 1$: $\quad \det A^{1\times 1} = a_{11}$

$n > 1$: $\quad \det A^{n\times n} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$

where $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is A after deletion of row i and column j

$A_{ij} = \left(A \right)$;

special case $n = 2$:

$$\det A^{2\times 2} = a_{11}\cdot a_{22} - a_{12}\, a_{21}$$

A matrix is "singular" if $\det A = 0$, "nonsingular" if $\det A \neq 0$

rules:

$$\det \mathbb{1}_n = 1$$
$$\det A^T = \det A$$
$$\det A\cdot B = \det A \cdot \det B$$
$$\det \operatorname{diag} d_{ii} = \prod_{i=1}^{n} d_{ii}$$
$$\det (a\cdot A) = a^n \det A$$

trace of a square matrix

another scalar characterizing a matrix

$$\operatorname{tr} A^{n\times n} = \operatorname{Sp} A^{n\times n} = \sum_{i=1}^{n} a_{ii}$$

inverse of a square matrix

nonsingular matrices have a unique inverse:
↳ what's that?

$$A\cdot A^{-1} = A^{-1}\cdot A = \mathbb{1}_n \qquad A \in \mathbb{R}^n,\ \det A \neq 0$$

rules:

$$\det A^{-1} = \frac{1}{\det A}$$ proove this!

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^{-1})^T = (A^T)^{-1}$$

Systems of linear equations

$$\begin{aligned}
(1)\quad & a_{11}\cdot x_1 + a_{12}\cdot x_2 + \dots + a_{1m}\cdot x_m = b_1\\
(2)\quad & a_{21}\cdot x_1 + a_{22}\cdot x_2 + \dots + a_{2m}\cdot x_m = b_2\\
\vdots\quad & \qquad\qquad \vdots\\
(n)\quad & a_{n1}\cdot x_1 + a_{n2}\cdot x_2 + \dots + a_{nm}\cdot x_m = b_n
\end{aligned}$$

rewrite this as matrix products:

$$A\cdot\vec{x} = \vec{b} \quad \text{with} \quad A\in\mathbb{R}^{n\times m},\ \vec{x}\in\mathbb{R}^m,\ \vec{b}\in\mathbb{R}^n$$

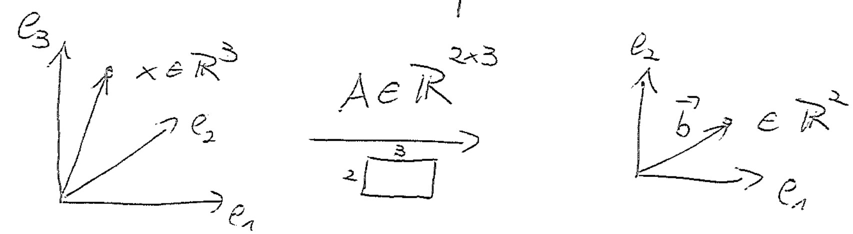
$(n\times m)$-matrices transform m-dim. vectors into n-dim. vectors

solve this equation = find solutions $\vec{x}$ for given A and $\vec{b}$

special case of square matrices:

if A^{-1} exists: $\vec{x} = A^{-1}\cdot A\vec{x} = A^{-1}\vec{b}$

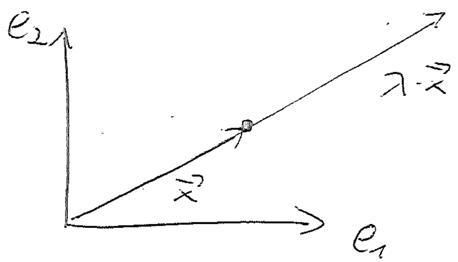
calculate determinant first!



eigenvector equations

$$A \cdot \vec{x} = \lambda \vec{x}$$

$A \in \mathbb{R}^{n \times n}$
$\vec{x} \in \mathbb{R}^{n}$
$\lambda \in \mathbb{R}$



action of A on $\vec{x}$
is a <u>scaling</u> only!

solve system of equations

$$(A - \lambda \mathbb{1}) \vec{x} = \vec{0}$$

orthogonal matrices

$$A^T \cdot A = A \cdot A^T = \mathbb{1}_n \iff A^T = A$$

orthogonal vectors via dot product:

$$\vec{x}_i \cdot \vec{x}_j = 0 \qquad \forall\, i \neq j$$

$$\vec{x}_i^{\,2} = 1 \qquad \text{„normalization"}$$